Determinacy, Learnability, and Plausibility in Monetary Policy Analysis: Additional Results

Bennett T. McCallum
Carnegie Mellon University

It is almost superfluous to begin by emphasizing that recent research in monetary policy analysis has featured a great deal of work concerning conditions for determinacy—that is, existence of a unique dynamically stable rational expectations equilibrium—under various specifications of policy behavior.[1] Indeed, there are a number of papers in which determinacy is the only criterion for a desirable monetary policy regime that is explicitly mentioned.[2]

By contrast, I have argued in recent publications (McCallum, 2003a, 2007) that least-squares (LS) learnability is a compelling necessary condition for a rational expectations (RE) equilibrium to be considered plausible, since individuals must somehow learn about the exact nature of an economy from data generated by that economy itself, while the LS learning process is biased toward a finding of learnability. A similar position has also been expressed by Bullard (2006). From such a position it follows that in conditions in which there is more than one dynamically stable RE solution—that is, indeterminacy—there may still be only one RE solution that is economically relevant, if the others are not LS learnable. In this sense, LS learnability is arguably a more important criterion than determinacy.

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1. Prominent examples include Benhabib et al. (2001), Clarida, Galí, and Gertler (1999), Rotemberg and Woodford (1999), Sims (1994), and Woodford (2003). Discussion in a leading textbook is provided by Walsh (2003).
2. See, for example, Carlstrom and Fuerst (2005). These authors would almost surely include other criteria if explicitly asked.

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It may be useful to expand briefly on the contention that LS learnability is a compelling necessary condition. The argument begins with the idea that, in actual economies, agents must ultimately obtain quantitative details concerning their economy, necessary for forming expectations, from data generated by that economy. Accordingly, the same should be true for the model economy used by a researcher. There are many conceivable learning processes, of course, so it would be rash to presume that any single one is relevant. Thus, it is not argued here that LS learnability is a sufficient condition for a RE equilibrium to be plausible. But the setup for LS learnability (see Evans and Honkapohja, 2003) is specified in a way that is, in a sense, biased towards a finding of learnability. Specifically, it assumes that agents know the correct structure qualitatively—that is, they know which variables are relevant. In addition, the process assumes that agents are collecting an ever-increasing number of observations on all relevant variables while the structure is remaining unchanged. Furthermore, the agents are estimating the relevant unknown parameters with an appropriate estimator.[3] Consequently, it seems, all in all, that if a proposed RE solution is not learnable by the LS process in question, it is implausible that it could prevail in practice.

Substantively, McCallum (2007) demonstrates that, in a very wide class of linear RE models, determinacy implies LS learnability (but not the converse) when individuals have knowledge of current conditions available for use in the learning process. This strong result does not pertain, however, if individuals have available, in the learning process, only information regarding previous values of endogenous variables.[4] One task of the present paper, accordingly, is to investigate the situation that is obtained when only lagged information is available. In addition, the paper will explore results that pertain when an alternative criterion of model plausibility, provisionally termed "well-formulated," characterizes the model's structure. In particular, it is shown that models that are well formulated, in the defined sense, often (but not invariably) possess the property of E-stability and hence LS learnability if current-period information is available in the learning process, even if determinacy does not prevail. Thus plausibility of a RE solution requires both that it be learnable and that the model at

3. A bit of additional discussion of the process is given below in section 2. Also see Evans and Honkapohja (2001, pp. 232-38).

4. Another limitation of the analysis of McCallum (2007) is that it considers only solutions of a form that excludes "resonant frequency sunspot" solutions. That limitation, which is maintained here, is discussed briefly in section 5.

hand be well formulated. A sufficient condition for both of these to hold, requiring that certain matrices have positive dominant diagonals, is introduced and considered below. Unfortunately, the situation in the case of lagged information is less favorable—that is, learnability can be assured only in special cases, for which no general characterization has been found.

1. Model and Determinacy

It will be useful to begin with a summary of the formulation and results developed in McCallum (2007). Throughout, we will work with a model of the form

$$\mathbf{y}_t = \mathbf{A}E_t\, \mathbf{y}_{t+1} + \mathbf{C}\mathbf{y}_{t-1} + \mathbf{D}\mathbf{u}_t, \tag{1}$$

where $\mathbf{y}_t$ is a $m\times 1$ vector of endogenous variables, $\mathbf{A}$ and $\mathbf{C}$ are $m\times m$ matrices of real numbers, $\mathbf{D}$ is $m\times n$, and $\mathbf{u}_t$ is a $n\times 1$ vector of exogenous variables generated by a dynamically stable process

$$\mathbf{u}_t = \mathbf{R}\mathbf{u}_{t-1} + \varepsilon_t, \tag{2}$$

with ε_t a white noise vector. It will not be assumed, even initially, that $\mathbf{A}$ is invertible. This specification is useful in part because it is the one utilized in Section 10.3 of Evans and Honkapohja (2001), for which E-stability conditions are reported on their p. 238.[5] Furthermore, the specification is very broad; in particular, any model satisfying the formulations of King and Watson (1998) or Klein (2000), can be written in this form—which will accommodate any number of lags, expectational leads, and lags of leads (see the appendix).

Following McCallum (1983, 1998), consider solutions to model (1)–(2) of the form

$$\mathbf{y}_t = \boldsymbol{\Omega}\mathbf{y}_{t-1} + \boldsymbol{\Gamma}\mathbf{u}_t, \tag{3}$$

in which $\boldsymbol{\Omega}$ is required to be real. Then, $E_t\mathbf{y}_{t+1} = \boldsymbol{\Omega}(\boldsymbol{\Omega}\mathbf{y}_{t-1} + \boldsymbol{\Gamma}\mathbf{u}_t) + \boldsymbol{\Gamma}\mathbf{R}\mathbf{u}_t$, and straightforward undetermined-coefficient reasoning shows that $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ must satisfy

5. Constant terms can be included in the equations of (1) by including an exogenous variable in $\mathbf{u}_t$ that is a random walk whose innovation has variance zero. In this case there is a borderline departure from process stability.

$$\mathbf{A}\boldsymbol{\Omega}^2 - \boldsymbol{\Omega} + \mathbf{C} = 0 \tag{4}$$

$$\boldsymbol{\Gamma} = \mathbf{A}\boldsymbol{\Omega}\boldsymbol{\Gamma} + \mathbf{A}\boldsymbol{\Gamma}\mathbf{R} + \mathbf{D}. \tag{5}$$

For any given $\boldsymbol{\Omega}$, equation (5) yields a unique $\boldsymbol{\Gamma}$ generically,[6] but there are many $m \times m$ matrices that solve (4) for $\boldsymbol{\Omega}$. Accordingly, the following analysis centers on equation (4). Since $\mathbf{A}$ is not assumed to be invertible, we write

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}^2 \\ \boldsymbol{\Omega} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{C} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega} \\ \mathbf{I} \end{bmatrix}, \tag{6}$$

in which the first row reproduces the matrix quadratic (4). Let the $2m \times 2m$ matrices on the left- and right-hand sides of equation (6) be denoted $\bar{\mathbf{A}}$ and $\bar{\mathbf{C}}$, respectively. Then, instead of focusing on the eigenvalues of $\bar{\mathbf{A}}^{-1}\bar{\mathbf{C}}$, which does not exist when $\mathbf{A}$ is singular, we solve for the (generalized) eigenvalues of the matrix pencil $(\bar{\mathbf{C}} - \lambda\bar{\mathbf{A}})$, alternatively termed the (generalized) eigenvalues of $\bar{\mathbf{C}}$ with respect to $\bar{\mathbf{A}}$ (see, for example, Uhlig, 1999). Thus, instead of diagonalizing $\bar{\mathbf{A}}^{-1}\bar{\mathbf{C}}$, as in Blanchard and Khan (1980), we use the Schur generalized decomposition, which serves the same purpose. Specifically, the Schur generalized decomposition theorem establishes that there exist unitary matrices $\mathbf{Q}$ and $\mathbf{Z}$ such that $\mathbf{Q}\,\bar{\mathbf{C}}\,\mathbf{Z} = \mathbf{T}$ and $\mathbf{Q}\,\bar{\mathbf{A}}\,\mathbf{Z} = \mathbf{S}$ with $\mathbf{T}$ and $\mathbf{S}$ triangular.[7] Then, eigenvalues of the matrix pencil $(\bar{\mathbf{C}} - \lambda\bar{\mathbf{A}})$ are defined as t_{ii}/s_{ii}. Some of these eigenvalues may be "infinite," in the sense that some s_{ii} may equal zero. This will be the case, indeed, whenever $\mathbf{A}$ and therefore $\bar{\mathbf{A}}$ are of less than full rank, since then $\mathbf{S}$ is also singular. All of the foregoing is true for any ordering of the eigenvalues and associated columns of $\mathbf{Z}$ (and rows of $\mathbf{Q}$). For the present, let us focus on the arrangement that places the t_{ii}/s_{ii} in order of decreasing modulus.[8]

6. Generically, $\mathbf{I}\text{-}\mathbf{R}' \otimes [(\mathbf{I} - \mathbf{A}\boldsymbol{\Omega})^{-1}\,\mathbf{A}]$ will be invertible, permitting solution of (5) for vec($\boldsymbol{\Gamma}$). Invertibility of $(\mathbf{I} - \mathbf{A}\boldsymbol{\Omega})$ is discussed in section 3.

7. Provided only that there exists some λ for which $\det[\bar{\mathbf{C}} - \lambda\bar{\mathbf{A}}] \neq 0$. See Klein (2000) or Golub and Van Loan (1996, p. 377). Note that in McCallum (2007) the matrices $\bar{\mathbf{A}}$ and $\mathbf{A}$ are denoted $\mathbf{A}$ and $\mathbf{A}_{11}$, respectively.

8. The discussion proceeds as if none of the t_{ii}/s_{ii} equals 1.0 exactly. If one does, the model can be adjusted, by multiplying some relevant coefficient by (for example) 0.9999.

To begin the analysis, pre-multiply equation (6) by $\mathbf{Q}$. Since $\mathbf{Q\bar{A}} = \mathbf{SH}$ and $\mathbf{Q\bar{C}} = \mathbf{TH}$, where $\mathbf{H} \equiv \mathbf{Z}^{-1}$, the resulting equation can be written as

$$\begin{bmatrix} \mathbf{S}_{11} & 0 \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{\Omega}^2 \\ \mathbf{\Omega} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{11} & \mathbf{0} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{I} \end{bmatrix}. \tag{7}$$

The first row of equation (7) reduces to

$$\mathbf{S_{11}(H_{11}\Omega + H_{12})\,\Omega = T_{11}(H_{11}\Omega + H_{12}).} \tag{8}$$

Then, if $\mathbf{H}_{11}$ is invertible, the latter can be used to solve for $\mathbf{\Omega}$ as

$$\mathbf{\Omega} = -\mathbf{H}_{11}{}^{-1}\,\mathbf{H}_{12} = -\mathbf{H}_{11}{}^{-1}(-\mathbf{H}_{11}\,\mathbf{Z}_{12}\,\mathbf{Z}_{22}{}^{-1}) = \mathbf{Z}_{12}\,\mathbf{Z}_{22}{}^{-1}, \tag{9}$$

where the second equality comes from the upper-right-hand submatrix of the identity $\mathbf{HZ} = \mathbf{I}$, provided that $\mathbf{H}_{11}$ is invertible, which is assumed without significant loss of generality.[9,10]

As mentioned above, there are many solutions $\mathbf{\Omega}$ to equation (4). These correspond to different arrangements of the eigenvalues, which result in different groupings of the columns of $\mathbf{Z}$ and therefore different compositions of the submatrices $\mathbf{Z}_{12}$ and $\mathbf{Z}_{22}$. Here, with the eigenvalues t_{ii}/s_{ii} arranged in order of decreasing modulus, the diagonal elements of $\mathbf{S}_{22}$ will all be non-zero, provided that $\mathbf{S}$ has at least m non-zero eigenvalues, which is assumed to be the case.[11] Clearly, for any solution under consideration to be dynamically stable, the eigenvalues of $\mathbf{\Omega}$ must be smaller than 1.0 in modulus. In McCallum (2007) it is shown that

$$\mathbf{\Omega} = \mathbf{Z}_{22}\,\mathbf{S}_{22}{}^{-1}\,\mathbf{T}_{22}\,\mathbf{Z}_{22}{}^{-1}, \tag{10}$$

9. This invertibility condition, also required by King and Watson (1998) and Klein (2000), obtains except in degenerate special cases of equation (1) that can be solved by simpler methods than considered here. Note that the invertibility of $\mathbf{H}_{11}$ implies the invertibility of $\mathbf{Z}_{22}$, given that $\mathbf{Z}$ and $\mathbf{H}$ are unitary.

10. Note that it is not being claimed that all solutions are of the form (9).

11. From its structure it is obvious that $\mathbf{\bar{A}}$ has at least m nonzero eigenvalues so, since $\mathbf{Q}$ and $\mathbf{Z}$ are nonsingular, $\mathbf{S}$ must have rank of at least m. This necessary condition is not sufficient for $\mathbf{S}$ to have at least m nonzero eigenvalues, however; hence the assumption.

so $\mathbf{\Omega}$ has the same eigenvalues as $\mathbf{S}_{22}{}^{-1}\,\mathbf{T}_{22}$. The latter is triangular, moreover, so the relevant eigenvalues are the m smallest of the $2m$ ratios t_{ii}/s_{ii} (given the decreasing-modulus ordering). For dynamic stability, the modulus of each of these ratios must then be less than 1. (In many cases, some of the m smallest moduli will equal zero.)

Let us henceforth refer to the solution under the decreasing-modulus ordering as the MOD solution. Now suppose that the MOD solution is stable. For it to be the only stable solution, there must be no other arrangement of the t_{ii}/s_{ii} that would result in a $\mathbf{\Omega}$ matrix with all eigenvalues smaller in modulus than 1.0. Thus, each of the t_{ii}/s_{ii} for $i = 1,\ldots, m$ must have modulus greater than 1.0, some perhaps infinite. Is there some $m \times m$ matrix whose eigenvalues relate cleanly to these ratios? Yes, it is the matrix $\mathbf{F} \equiv (\mathbf{I} - \mathbf{A\Omega})^{-1}\mathbf{A}$, which appears frequently in the analysis of Binder and Pesaran (1995, 1997).[12] Regarding this $\mathbf{F}$ matrix, it is shown that, for any ordering such that $\mathbf{H}_{11}$ is invertible, including the MOD ordering, we have the equality

$$\mathbf{H}_{11}\mathbf{F}\,\mathbf{H}_{11}{}^{-1} = \mathbf{T}_{11}{}^{-1}\mathbf{S}_{11}, \tag{11}$$

which implies that $\mathbf{F}$ has the same eigenvalues as $\mathbf{T}_{11}{}^{-1}\mathbf{S}_{11}$. In other words, it is the case that the eigenvalues of $\mathbf{F}$ are the same, for any given arrangement of the system's eigenvalues, as the *inverses* of the values of t_{ii}/s_{ii} for $i = 1, \ldots, m$. Under the MOD ordering, these are the inverses of the first (largest) m of the eigenvalues of the system's matrix pencil. Accordingly, for solution (9) to be the only stable solution, all the eigenvalues of the corresponding $\mathbf{F}$ must be smaller than 1.0 in modulus. This result, stated in different ways, is well known from Binder and Pesaran (1995), King and Watson (1998), and Klein (2000), and is an important generalization of one result of Blanchard and Khan (1980) for a model with nonsingular $\mathbf{A}$.

Thus we have established notation for models of form (1)–(2) and have reported results showing that the existence of a unique stable solution requires that all eigenvalues of the defined $\mathbf{\Omega}$ matrix and the corresponding $\mathbf{F}$ be less than 1.0 in modulus. It will be convenient to express that condition as follows: all $|\lambda_{\mathbf{\Omega}}| < 1$ and all $|\lambda_{\mathbf{F}}| < 1$.

12. There is no general proof of invertibility of $[\mathbf{I} - \mathbf{A\Omega}]$, but if $\mathbf{A\Omega}$ were by chance to have some eigenvalue exactly equal to 1.0, that condition could be eliminated by making some small adjustment to elements of $\mathbf{A}$ or $\mathbf{C}$. Also, see section 4 below.

2. E-Stability in Two Cases

Let us now turn to conditions for learnability under two different information assumptions. First we will review the main results from McCallum (2007), which assumes that agents have full information on current values of endogenous variables during the learning process, and then we will go on to the second assumption, namely, that only lagged values of endogenous variables are known during the learning process. The manner in which learning takes place in Evans and Honkapohja's analysis is as follows. Agents are assumed to know the structure of the economy as specified in equations (1) and (2), in the sense that they know what variables are included, but do not know the numerical values of the parameters. What they need to know, to form expectations, is values of the parameters of the solution equations (3). In each period t, they form forecasts on the basis of a least squares regression of the variables in $\mathbf{y}_{t-1}$ on previous values of $\mathbf{y}_{t-2}$ and any exogenous observables. Given those regression estimates, however, expectations of $\mathbf{y}_{t+1}$ may be calculated assuming knowledge of $\mathbf{y}_t$ or, alternatively, assuming that $\mathbf{y}_{t-1}$ is the most recent observation possessed by agents and is thus usable in the forecasting process. In the former case, the conditions for E-stability reported by Evans and Honkapohja (2001) are that the following three matrices must have all eigenvalues with real parts less than 1.0:

$$\mathbf{F} \equiv (\mathbf{I} - \mathbf{A\Omega})^{-1}\mathbf{A}, \tag{12a}$$

$$\left[(\mathbf{I}\text{-}\mathbf{A\Omega})^{-1}\mathbf{C}\right]' \otimes \mathbf{F}, \tag{12b}$$

$$\mathbf{R}' \otimes \mathbf{F}. \tag{12c}$$

In the second case, however, the analogous condition (Evans and Honkapohja, 2001) is that the following matrices must have all eigenvalues with real parts less than 1.0:

$$\mathbf{A}\,(\mathbf{I} + \mathbf{\Omega}), \tag{13a}$$

$$\mathbf{\Omega}' \otimes \mathbf{A} + \mathbf{I} \otimes \mathbf{A\Omega}, \tag{13b}$$

$$\mathbf{R}' \otimes \mathbf{A} + \mathbf{I} \otimes \mathbf{A\Omega}. \tag{13c}$$

Except in the case that $\mathbf{\Omega} = \mathbf{0}$, which will result when $\mathbf{C} = \mathbf{0}$, these conditions are not equivalent to those in equation (12).

It is important to note that use of the first information assumption is not inconsistent with a model specification in which supply and demand decisions in period t are based on expectations formed in the past, such as $E_{t-1}\mathbf{y}_{t+j}$ or $E_{t-2}\mathbf{y}_{t+j}$. It might also be mentioned parenthetically that conditions (12) and (13) literally pertain to the E-stability of the model (1)–(2) under the two information assumptions, not its learnability. Under quite broad conditions, however, E-stability is necessary and sufficient for LS learnability. This near-equivalence is referred to by Evans and Honkapohja as the "E-stability principle" (Evans and Honkapohja, 1999, 2001). Since E-stability is technically easier to verify, applied analysis typically focuses on it, rather than on direct exploration of learnability.

Given the foregoing discussion, it is a simple matter to verify that if a model of form (1)–(2) is determinate, then it satisfies conditions (12). First, determinacy requires that all eigenvalues of $\mathbf{F}$ have modulus less than 1.0, so their real parts must all be less than 1.0, thereby satisfying (12a). Second, from equation (4) it can be seen that $(\mathbf{I}–\mathbf{A\Omega})^{-1}\mathbf{C} = \mathbf{\Omega}$. Therefore, matrix (12b) can be written as $\mathbf{\Omega}' \otimes \mathbf{F}$. Furthermore, it is a standard result (Magnus and Neudecker, 1988) that the eigenvalues of a Kronecker product are the products of the eigenvalues of the relevant matrices (for example, the eigenvalues of $\mathbf{\Omega}' \otimes \mathbf{F}$ are the products $\lambda_{\mathbf{\Omega}}\, \lambda_{\mathbf{F}}$). Therefore, condition (12b) holds. Finally, since $|\lambda_{\mathbf{F}}| < 1$, condition (12c) holds provided that all $|\lambda_{\mathbf{R}}| \leq 1$, which has been assumed by specifying that equation (2) is dynamically stable.

Determinacy does not imply learnability, however, under the second information assumption. This point, which is developed by Evans and Honkapohja (2001), can be illustrated by means of a bivariate example.[13] Let the $\mathbf{y}_t$ vector in equation (1) include two variables, y_{1t} and y_{2t}, related by the dynamic model that follows:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} -0.01 & 0.01 \\ 0.99 & -0.01 \end{bmatrix} \begin{bmatrix} E_t y_{1t+1} \\ E_t y_{2t+1} \end{bmatrix} + \begin{bmatrix} 0.02 & 1.10 \\ 0.01 & 0.06 \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} \qquad (14)$$

13. Its specification is close numerically to the qualitative version of the Evans and Honkapohja example that is used in McCallum (2007), pp. 1386–88.

Then, for the MOD solution we have

$$\mathbf{A\Omega} = \begin{bmatrix} -0.01 & 0.01 \\ 0.99 & -0.01 \end{bmatrix} \begin{bmatrix} 0.0218 & 1.1133 \\ -0.095 & -0.774 \end{bmatrix} = \begin{bmatrix} -0.0012 & -0.0189 \\ 0.0225 & 1.1099 \end{bmatrix}, \quad (15)$$

with eigenvalues of $\mathbf{\Omega}$ being –0.148 and –0.604, while

$$\mathrm{F} = \begin{bmatrix} 0.1604 & 0.00831 \\ -9.040 & 0.0893 \end{bmatrix},$$

which has (complex) eigenvalues $0.1249 \pm 0.2717\, i$. Inspection of these shows that this solution is determinate, and that conditions (12a) and (12b), relevant for E-stability in the case in which current information is available during learning, are satisfied. Let us assume $\mathbf{R} = \mathbf{0}$, that is, white noise disturbances, for simplicity. Then the determinate RE solution is E-stable and learnable under the first information assumption.

But for the case with only lagged information during learning, it is necessary to consider the eigenvalues of the matrices shown in expressions (13). For equation (13a), the matrix $\mathbf{A}(\mathbf{I} + \mathbf{\Omega})$ is

$$\begin{bmatrix} -0.0112 & -0.0089 \\ 1.0125 & 1.0999 \end{bmatrix}$$

whose eigenvalues are -0.0030 and 1.0918. The last of these violates the condition for equation (13a), however, so under the lagged-information assumption, the relevant E-stability condition is not satisfied and the determinate RE equilibrium is not LS learnable.

This result exemplifies the fact that determinacy is not generally sufficient for learnability of RE solutions, although it is sufficient under the first information assumption. Of equal importance, in my opinion, is the fact that determinacy is not necessary for learnability. In particular, the MOD solution can be learnable, and be the only learnable solution of form (3), in cases in which indeterminacy prevails. One such example is given in McCallum (2007).[14] In such cases, the

14. I take this opportunity to point out that McCallum (2007, p. 1386), errs in stating that when the eigenvalues are … “30.65, −0.532, −0.123, and 0.000 … both stable solutions are learnable.” Actually, only the MOD solution is learnable.

position that learnability is necessary for a solution to be plausible would suggest that there may be no problem implied by the absence of determinacy.[15]

3. Well-Formulated Models

McCallum (2003b) suggests that there is a distinct and neglected property that dynamic models should possess to be considered "well-formulated" and plausible for the purposes of economic analysis. To begin the discussion, consider first the single-variable case of specification (1),

$$y_t = \alpha E_t y_{t+1} + c y_{t-1} + u_t, \tag{16}$$

with $u_t = (1 - \rho)\eta + \rho u_{t-1} + w_{t,}$ with $|\rho| < 1$ and w_t white noise. Thus, u_t is an exogenous forcing variable with an unconditional mean of η (assumed nonzero) and units have been chosen so that there is no constant term. Applying the unconditional expectation operator to equation (16) yields

$$Ey_t = \alpha Ey_{t+1} + cEy_{t-1} + \eta. \tag{17}$$

In this case, y_t will be covariance stationary, and we have

$$Ey_t = \frac{\eta}{[1-(a+c)]}. \tag{18}$$

But from the latter, it is clear that as $a + c$ approaches 1.0 from above, the unconditional mean of y_t approaches $-\infty$ (assuming, without loss of generality, that $\eta > 0$), whereas if $a + c$ approaches 1.0 from below, the unconditional mean approaches $+\infty$. Thus, there is an infinite discontinuity at $a + c = 1.0$. This implies that a tiny change in $a + c$ could alter the average (that is, steady–state) value Ey_t from an arbitrarily large positive number to an arbitrarily large negative number. Such a property seems highly implausible and therefore unacceptable for a well-formulated model.[16] The

15. Disregarding, that is, "sunspot" solutions not of form (3).

16. The model could be formulated with the exogenous variable also written in terms of percent or fractional deviations from the reference level η, for example, $\hat{u}_t = u_t - \eta$. But that would not alter the relationship between Ey_t and η, which can be extremely sensitive to tiny changes in $a + c$.

substantive problem is not eliminated, obviously, by adoption of the zero-measure exclusion $a + c \neq 1$.

In light of the foregoing observation, it is my contention that, to be considered well formulated (WF), the model at hand needs to include a restriction on its admissible parameter values; a restriction that rules out $a + c = 1$ and yet admits a large interval of values that includes $(a,c) = (0,0)$. In the case at hand, the appropriate restriction is $a + c < 1$. Of course, $a + c > 1$ would serve just as well mathematically to avoid the infinite discontinuity, but it seems clear that $a + c < 1$ is vastly more appropriate from an economic perspective since it includes the values (0,0)[17] Since we want this condition to apply to $a + c$ sums between zero and that value that pertains to the model at hand, our requirement for WF is that a and c satisfy $1 - \varepsilon\,(a + c) > 0$ for all $0 \leq \varepsilon \leq 1$. [It should be clear, in addition, that the foregoing argument could be easily modified to apply to y_t processes that are trend stationary, rather than strictly (covariance) stationary.] It is shown in McCallum (2003b) that under this requirement, plus a second one to be discussed shortly, the univariate model (16) is invariably E-stable.[18]

Next, for the bivariate case of model (1), extension of the foregoing WF property requires that $\mathbf{A}$ and $\mathbf{C}$ be such that $\det[\mathbf{I} - \varepsilon(\mathbf{A} + \mathbf{C})]$ is positive for all $0 \leq \varepsilon \leq 1$; otherwise, the steady-state values of the variables may possess infinite discontinuities. But there are other requirements as well. Let ac_{ij} temporarily denote the ij^{th} element of $\mathbf{A} + \mathbf{C}$. Then the model with $y_1 = Ey_{1t}$, $y_2 = Ey_{2t}$, $\eta_1 = Eu_{1t}$ and $\eta_2 = Eu_{2t}$ implies

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} ac_{11} & ac_{12} \\ ac_{21} & ac_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{19}$$

so that $E\mathbf{y} = [\mathbf{I} - (\mathbf{A} + \mathbf{C})]^{-1}\,\boldsymbol{\eta}$ can be written as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} 1 - ac_{22} & ac_{12} \\ ac_{21} & 1 - ac_{11} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{20}$$

17. In models of the linear form (16), one would expect coefficients a and c typically to represent elasticities and often to be numerically small relative to 1.

18. That paper's analysis of multivariate systems is, however, unsatisfactory.

where $\Delta = \det[\mathbf{I} - (\mathbf{A} + \mathbf{C})] = (1 - ac_{11})(1 - ac_{22}) - ac_{12}\, ac_{21}$. Then the counterpart of the univariate requirement that $1 - \varepsilon\,(a + c) > 0$ includes the condition $\Delta > 0$ [for all $0 \leq \varepsilon \leq 1$].[19] We must rule out, however, the case in which $\Delta > 0$ results from $1 - \varepsilon ac_{11}$ and $1 - \varepsilon ac_{22}$ both being negative.[20] The condition on Δ should be extended, therefore, to also require $1 - \varepsilon ac_{11} > 0$ and $1 - \varepsilon ac_{22} > 0$.

How are these WF requirements extended to pertain to cases with more than two variables? It appears that the appropriate requirement is that $[\mathbf{I} - \varepsilon(\mathbf{A} + \mathbf{C})]$ be a P-matrix, which by definition has all its principal minors positive and thereby imposes the conditions discussed for the cases above in which m equals 1 and 2. Other properties of any P-matrix are that its inverse exists and is itself a P-matrix, and that all its real eigenvalues are positive.[21]

An alternative possibility that is of interest would be to require $[\mathbf{I} - \varepsilon(\mathbf{A} + \mathbf{C})]$ to be a positive dominant-diagonal matrix.[22] This requirement would have implications for the E-stability status of the model, as will be discussed below, and positive dominant-diagonal (PDD) matrices have an important tradition in dynamic economics stemming from the literature on multimarket stability analysis. This condition is, however, somewhat stronger than is actually required by our objective of ruling out specifications in which leading implications of the model are hyper-sensitive to parameter values.

As a brief but relevant digression, one example of a matrix that is a P-matrix and yet is not positive dominant-diagonal is as follows:

$$\begin{bmatrix} 0.08 & -0.92 & 0.90 \\ 0.92 & 0.07 & -0.03 \\ -0.72 & 0.30 & 0.04 \end{bmatrix}. \tag{21}$$

Clearly, the entries in any row show immediately that this matrix is not positive dominant diagonal (PDD). But its determinant is 0.3087 and the three second-order minors are 0.0118, 0.651, and 0.852. Since the diagonal elements are also all positive, the matrix

19. Henceforth the bracketed condition is to be understood wherever relevant.
20. This is clear for the case in which $\mathbf{A} + \mathbf{C}$ is a diagonal matrix.
21. On the topic of P-matrices, see Horn and Johnson (1991) and Gale and Nikaido (1965).
22. Again, see Horn and Johnson (1991) and Gale and Nikaido (1965).

is a P-matrix. For future reference, note that its eigenvalues are –0.0067 + 1.2319*i*, –0.0067 – 1.2319*i*, and 0.2034. Thus the example illustrates the fact that, although a P-matrix cannot have a negative real eigenvalue, it can have a complex eigenvalue pair with negative real parts.[23]

Returning now to the main line of argument, there is a second type of discontinuity that should also be eliminated for a model to be viewed as WF, namely, infinite discontinuities in its impulse response functions. In model (1)–(2) with solution (3), the impulse response to the shock vector $\mathbf{u}_t$ involves the matrix $\mathbf{\Gamma}$, which is given by

$$\mathbf{\Gamma} = \mathbf{A\Omega\Gamma} + \mathbf{A\Gamma R} + \mathbf{D}. \tag{22}$$

Thus, $(\mathbf{I} - \mathbf{A\Omega})\,\mathbf{\Gamma} = \mathbf{A\Gamma R} + \mathbf{D}$ so using $\mathbf{F} = (\mathbf{I} - \mathbf{A\Omega})^{-1}\mathbf{A}$, equation (22) can be written as

$$\mathbf{\Gamma} = \mathbf{F\Gamma R} + (\mathbf{I} - \mathbf{A\Omega})^{-1}\mathbf{D}. \tag{23}$$

Then, using the well-known identity that, for any conformable matrix product $\mathbf{ABC}$ it is true that vec $\mathbf{ABC} = (\mathbf{C}' \otimes \mathbf{A})$ vec $\mathbf{B}$,[24] it follows that

$$\mathrm{vec}\mathbf{\Gamma} = (\mathbf{R}' \otimes \mathbf{F})\mathrm{vec}\mathbf{\Gamma} + \mathrm{vec}\left[(\mathbf{I} - \mathbf{A\Omega})^{-1}\mathbf{D}\right] \tag{24}$$

implying

$$\mathrm{vec}\mathbf{\Gamma} = \left[\mathbf{I} - (\mathbf{R}' \otimes \mathbf{F})\right]^{-1}\mathrm{vec}\left[(\mathbf{I} - \mathbf{A\Omega})^{-1}\mathbf{D}\right]. \tag{25}$$

Accordingly, our second WF requirement is for $[\mathbf{I} - (\mathbf{R}' \otimes \mathbf{F})]$ and $(\mathbf{I} - \mathbf{A\Omega})$ to be well behaved in the same manner as $\mathbf{I} - (\mathbf{A} + \mathbf{C})$, that is, that each is a P-matrix. Again it is of interest to consider the possibility of requiring that each of these be a PDD matrix.

23. See Horn and Johnson (1991, p. 123).

24. See, for example, Evans and Honkapohja (2001, p. 117) or Magnus and Neudecker (1988, p. 28).

4. E-Stability in WF Models?

In this section, the concern is with the relationship between models that are WF and those in which the MOD solution is learnable. That there may be some significant relationship is suggested by the following identity:

$$(\mathbf{I} - \mathbf{A\Omega})(\mathbf{I} - \mathbf{F})(\mathbf{I} - \mathbf{\Omega}) = \mathbf{I} - (\mathbf{A} + \mathbf{C}), \tag{26}$$

which is mentioned by Binder and Pesaran (1995).[25] From this equation, it is clear that that non-singularity of $\mathbf{I} - (\mathbf{A} + \mathbf{C})$ implies that the three matrices $(\mathbf{I} - \mathbf{A\Omega})$, $(\mathbf{I} - \mathbf{F})$ and $(\mathbf{I} - \mathbf{\Omega})$ are all nonsingular. In addition, we can see that the WF requirement that $\det[\mathbf{I} - \varepsilon(\mathbf{A} + \mathbf{C})]$ is positive for all $0 \leq \varepsilon \leq 1$ also implies that the real eigenvalues of $\mathbf{\Omega}$, $\mathbf{A\Omega}$, and $\mathbf{F}$ must all be less than 1.0 in value.[26] To make that argument, consider the situation when $\mathbf{A}$ and $\mathbf{C}$ are multiplied by ε, $0 \leq \varepsilon \leq 1$. For very small values of ε, the matrices $\mathbf{\Omega}$, $\mathbf{A\Omega}$, and $\mathbf{F}$ will all be small so the eigenvalues of all four matrices in equation (26) will be close to 1.0 and their determinants will be positive. Now let ε increase and approach 1.0. If $\mathbf{I} - \varepsilon(\mathbf{A} + \mathbf{C})$ remains nonsingular throughout this process, so too will each of the three matrices on the left-hand side of equation (26). Since a real eigenvalue of zero would imply singularity for any of the matrices in question, and since eigenvalues are continuous functions of the matrix elements, the stated result is valid.

Accordingly, the WF requirement that $\det[\mathbf{I} - \varepsilon(\mathbf{A} + \mathbf{C})]$ is positive for all $0 \leq \varepsilon \leq 1$ also implies that the real eigenvalues of $\mathbf{\Omega}$, $\mathbf{A\Omega}$, and $\mathbf{F}$ are all less than 1.0 in value. In addition, the requirement that the matrix $[\mathbf{I} - (\mathbf{R}' \otimes \mathbf{F})]$ be a P-matrix implies that all the real eigenvalues of $(\mathbf{R}' \otimes \mathbf{F})$ will be smaller than 1.0. Therefore, condition (12c), as well as (12a), is satisfied. What about the remaining condition, for the current-information case, (12b)? Here we recognize that, by rearrangement of equation (4), $(\mathbf{I} - \mathbf{A\Omega})^{-1}\mathbf{C} = \mathbf{\Omega}$. Accordingly, condition (12b) becomes $\mathbf{\Omega}' \otimes \mathbf{F}$. But then note that with the MOD ordering it is the case that all $|\lambda_{\mathbf{\Omega}}| < 1/|\lambda_{\mathbf{F}}|$ so all $|\lambda_{\mathbf{\Omega}}||\lambda_{\mathbf{F}}| < 1$. But $|\lambda_{\mathbf{\Omega}}||\lambda_{\mathbf{F}}| = |\lambda_{\mathbf{\Omega}}\lambda_{\mathbf{F}}| \geq \mathrm{Re}(\lambda_{\mathbf{\Omega}}\lambda_{\mathbf{F}})$ so it follows that this condition is invariably satisfied. Accordingly,

25. The identity can be verified by writing out $\mathbf{F}$ in the left side of equation (26), multiplying, cancelling, and inserting $\mathbf{C}$ for $\mathbf{\Omega} - \mathbf{A\Omega}^2$.

26. Here, and often in what follows, I use the fact that the eigenvalues of a matrix of form $(\mathbf{I} - \mathbf{B})$ satisfy $\lambda_{\mathbf{I-B}} = 1 - \lambda_{\mathbf{B}}$.

with current information available during the learning process, the MOD solution would be learnable, when the model is WF, if all eigenvalues were real.

Unfortunately, there is no guarantee that the real part of all complex eigenvalues will be smaller than 1.0. The situation is described by Horn and Johnson (1991) as follows: "if **A** is a n–by–n P-matrix ... then every eigenvalue of **A** lies in the open angular wedge $\mathbf{W}_n \equiv \{z = re^{i\theta}: |\theta| < \pi - (\pi/n),\ r > 0\}$. Moreover, every point in $\mathbf{W}_n$ is an eigenvalue of some n-by-n P-matrix." But for $n > 2$, $\mathbf{W}_n$ includes points in the in the two left-hand quadrants in the complex plane. Therefore, it cannot be argued that, in general, the WF condition implies LS learnability for the MOD solution.

In this regard, note that, since **A** and **C** are matrices of real numbers, $\mathbf{I} - (\mathbf{A} + \mathbf{C})$ will have only real eigenvalues if $\mathbf{A} + \mathbf{C}$ is symmetric. And since eigenvalues are continuous functions of the elements of the matrix in question, these eigenvalues will be real if $\mathbf{A} + \mathbf{C}$ does not depart too far from symmetry. Diagonal matrices are of course symmetric, so it is not surprising that dominant-diagonal matrices have strong properties pertaining to their eigenvalues. In particular, if a real matrix is positive diagonal dominant (PDD), that is, is diagonal-dominant with all diagonal elements positive, then all its eigenvalues will have positive real parts—see Horn and Johnson (1985). Accordingly, if we were to require (as mentioned above) that $\mathbf{I} - (\mathbf{A} + \mathbf{C})$, $(\mathbf{I} - \mathbf{A}\boldsymbol{\Omega})$, and $[\mathbf{I} - (\mathbf{R}' \otimes \mathbf{F})]$ were PDD, rather than just P-matrices, then learnability would be implied. That possibility is not, however, justified by the line of argument used to motivate the WF condition, that is, by the desirability of ruling out infinite discontinuities in impulse response functions (and the model's steady-state values).

The argument, then, is that being WF is an additional, distinct, plausibility condition to be required along with learnability. Only if a RE solution is both learnable, and results from a model that is WF, would it be considered as a plausible candidate for a RE solution that might prevail in reality. This may seem like a rather demanding requirement. But most realistic models utilized in monetary policy analysis easily meet both of these conditions; difficulties arise primarily in the case of zero-lower-bound situations, very strong policy responses to expected future conditions, and other extreme conditions.

In any case, the potential attractiveness of the WF requirement, in addition to that of LS learnability, is exemplified by an example considered for other purposes in McCallum (2004). The example in table 2 of that paper combines two univariate models of form

(1)–(2), one of which has two explosive solutions and the other of which has two stable solutions.[27] Small off-diagonal elements of the **A** and **C** matrices are added to make the combined model a bivariate example that is not reducible (while barely changing the system eigenvalues). In this bivariate model it is found that there is a unique stable solution.[28] Under the current-information assumption, then, this equilibrium is learnable as well as determinate. It hardly seems plausible, however, to believe that the combination of an explosive sector plus an indeterminate sector, with only minimal interaction between them, would result in overall behavior reflecting a well-behaved, unique equilibrium. Thus the finding that the determinate and learnable solution pertains to a model that is not well-formulated, is highly relevant and leads to a conclusion that seems entirely sensible.[29] The appropriate conclusion is that this solution is not plausible. The other solution (of form (3)) is the MSV solution. It is learnable but not dynamically stable.[30] Thus the conclusion of an analysis based on the requirement that a plausible RE equilibrium must be stable, learnable, and WF is that the system under discussion has no such equilibrium. That seems eminently sensible, for a model that is the combination of one explosive sector and one indeterminate sector with very little interaction.

Next we consider learnability for WF models under the second information assumption, for which the relevant conditions are that all eigenvalues of the matrices in conditions (13a)–(13c) have real parts less than 1.0. Let us assume that $\mathbf{I} - (\mathbf{A} + \mathbf{C})$, $(\mathbf{I} - \mathbf{A\Omega})$ and $[\mathbf{I} - (\mathbf{R}' \otimes \mathbf{F})]$ are all PDD matrices, which makes the MOD solution both learnable and WF. First consider condition (13a), which implies that $\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{\Omega})$ must have all eigenvalues with real parts that are positive. Using the definition of **F**, we can write

$$(\mathbf{I} - \mathbf{A\Omega})(\mathbf{I} - \mathbf{F}) = (\mathbf{I} - \mathbf{A\Omega})\,[\mathbf{I} - (\mathbf{I} - \mathbf{A\Omega})^{-1}\mathbf{A}] = (\mathbf{I} - \mathbf{A\Omega}) - \mathbf{A} = \mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{\Omega}). \quad (27)$$

27. Incidentally, in that paper's equation (29), the lower-left element of **C** is 0.3, not 0.5.

28. Which differs from the minimum-state-variable (MSV) solution in the sense of McCallum (2003b).

29. The non-WF conclusion is based on violations of both steady-state and impulse response requirements. For the other solution of form (3), the steady-state WF conditions are violated.

30. For learning of explosive solutions, a modified condition pertaining to shock variances is required. See Evans and Honkapohja (2001, pp. 219–20).

Now, our discussion above indicates that $\mathbf{I} - \mathbf{A\Omega}$ and $\mathbf{I} - \mathbf{F}$ will both have eigenvalues with all real parts positive under the WF assumption, so equation (27) indicates that this property would carry over to $\mathbf{I} - \mathbf{A}(\mathbf{I} + \mathbf{\Omega})$. This would not be the case, however, if the only specification is that $\mathbf{I} - (\mathbf{A} + \mathbf{C})$, $(\mathbf{I} - \mathbf{A\Omega})$ and $[\mathbf{I} - (\mathbf{R}' \otimes \mathbf{F})]$ are P-matrices.

Even in the more favorable case, with PDD matrices, no general results pertaining to conditions (13b) and (13c) have been found. The problem is that sums of Kronecker products do not in general yield matrices for which eigenvalues are cleanly related to those of the individual matrices. Nevertheless, there are two special cases that can be treated readily. First, consider the case in which $\mathbf{C} = \mathbf{0}$, so there are no predetermined variables in the solution, which implies that $\mathbf{\Omega} = \mathbf{0}$. Then, $\mathbf{F} = (\mathbf{I} - \mathbf{A\Omega})^{-1}\mathbf{A} = \mathbf{A}$, and thus condition (13a) becomes the same as (12a). Furthermore, (13b) is irrelevant with $\mathbf{\Omega} = \mathbf{0}$ and (13c) becomes $(\mathbf{R}' \otimes \mathbf{A})$, which is the same as in (12c). So in this case, the two information assumptions yield the same E-stability conditions. Second, suppose that $\mathbf{C} \neq \mathbf{0}$, but that the exogenous variables are white noise, that is, $\mathbf{R} = \mathbf{0}$. Then condition (13c) becomes $(\mathbf{I} \otimes \mathbf{A\Omega})$ and the result based on $(\mathbf{I} - \mathbf{A\Omega})^{-1}$ shows that this condition will be satisfied if the latter matrix is PDD. But conditions pertaining to (13a) and (13b) are not necessarily satisfied. Of course, it is a simple matter to examine specific cases numerically.

5. General Issues

A number of possible objections to the foregoing argument need to be addressed. Probably the most prominent among researchers in the area would be the fact that our analysis has been concerned only with solutions of form (3), which excludes sunspot solutions of the "resonant frequency" type. It is my position, however, that the learning process pertaining to solutions of this type is much less plausible than for solutions of form (3). In particular, the solutions are not of the standard vector-autoregression (VAR) form. Therefore, an agent who experimented with many different specifications of VAR models, using the economy's generated time series data, would still not be led to such a solution. Indeed, it seems to me that arguments suggesting that that type of learning could exist in actual economies are utterly implausible. Of course, literally speaking, RE itself is implausible—as early critics emphasized. Nevertheless, RE is rightly regarded by mainstream researchers as the appropriate assumption for economic analysis, especially policy analysis. That is the case because RE is fundamentally the assumption that agents optimize with respect to their expectational behavior, just as they do (according to neoclassical economic analysis) with respect to other basic economic activities such as selection of consumption bundles, selection of quantities produced and inputs utilized, etc.—for a necessary condition for optimization is that individuals eliminate any systematically erroneous component of their expectational behavior. Also, RE is doubly attractive (to researchers) from a policy perspective, for it assures that a researcher does not propose policy rules that rely upon policy behavior that is designed to exploit patterns of suboptimal expectational behavior by individuals.

Another issue is the possible use of learning behavior not as a device for assessing the plausibility of rational expectations equilibria, but as a replacement for the latter. This type of approach is discussed by Evans and Honkapohja (2001) and has been prominent in the work of Orphanides and Williams (2005), among others. Use of constant-gain learning (Evans and Honkapohja, 2001) provides a sensible alternative to the decreasing-gain learning implicit in the LS learning/E-stability literature. This approach, however, does not seem to solve the "startup" problem, that is, the issue of how the economy will behave in the first several periods following the adoption of a new policy rule or the occurrence of some other structural change. It is highly unlikely that economies will move promptly to new RE equilibria following such a change, and I doubt that they would move promptly to a modeled learning path. In both cases, I share the opinion voiced by Lucas (1980), to the effect that, after a structural change (including policy regime changes), reliable analysis should pertain to the economy's behavior after it has had time to settle into a new dynamic stochastic equilibrium.

6. Conclusion

Let us now conclude with a very brief review of the points developed above. First, the paper reviews a previous result to the effect that, under the "first" information assumption that agents possess knowledge of current endogenous variables in the learning process, determinacy of a RE equilibrium is a sufficient but not necessary condition for least-squares learnability of that equilibrium. Thus, since learnability is an attractive necessary condition for plausibility of any equilibrium, there may exist a single plausible RE solution even in cases of indeterminacy. In addition, the paper proposes and outlines a distinct criterion that plausible models should possess, termed "well formulated" (WF), that rules out infinite discontinuities in the model's implied steady–state values of endogenous variables and in its impulse response functions. The paper then explores the relationship between this WF property and learnability, under the first information assumption, and finds that (although they often agree) neither implies the other. Extending the P-matrix requirement, implied for specified matrices by the WF property, to one that demands positive dominant-diagonal matrices would guarantee both WF and learnability, but a suitable rationale for such a requirement has not been found. Finally, under the second information assumption, which gives the agents only lagged information on endogenous variables during the learning process, the situation is less favorable in the sense that learnability can be guaranteed only under special assumptions.

Appendix

To demonstrate that a very wide variety of linear RE models can be written in form (1)–(2), consider the formulation of King and Watson (1998) or Klein (2000), as exposited by McCallum (1998), as follows:

$$\begin{bmatrix} \mathbf{A}_{11}^{*} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} E_t \mathbf{x}_{t+1} \\ \mathbf{k}_{t+1} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{k}_t \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} [\mathbf{v}_t]. \tag{A1}$$

Here $\mathbf{v}_t$ is an AR(1) vector of exogenous variables (including shocks) with stable AR matrix $\mathbf{R}$, while $\mathbf{x}_t$ and $\mathbf{k}_t$ are $m_1 \times 1$ and $m_2 \times 1$ vectors of non-predetermined and predetermined endogenous variables, respectively. It is assumed, without significant loss of generality, that $\mathbf{B}_{11}$ is invertible[31] and that $\mathbf{G}_2 = 0$.[32] Define $\mathbf{y}_t = [\mathbf{x}_t{}' \quad \mathbf{k}_t{}' \quad \mathbf{x}_{t-1}{}' \quad \mathbf{k}_{t-1}{}']'$ and write the system in form (1) with $\mathbf{u}_t = \mathbf{v}_t$ and the matrices given, as follows:

$$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{B}_{11}^{-1}\mathbf{A}_{11}^{*} & 0 & 0 & -\mathbf{B}_{11}^{-1}\mathbf{B}_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \mathbf{B}_{21}\mathbf{B}_{22} & 0 & 0 & 0 \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} -\mathbf{B}_{11}^{-1}\mathbf{G} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{A2}$$

This representation is important because it is well known that the system (A1) permits, via use of auxiliary variables, any finite number of lags, expectational leads, and lags of expectational leads for the basic endogenous variables. Also, any higher-order AR process for the exogenous variables can be written in AR(1) form.[33] Thus it has been shown that the Evans and Honkapohja (2001) formulation is in fact rather general, although it does not pertain to asymmetric information models.

31. For the system (A1) to be cogent, each of the m_1 non-predetermined variables must appear in at least one of the m_1 equations of the first matrix row. Then the diagonal elements of $\mathbf{B}_{11}$ will all be non-zero and to avoid inconsistencies the rows of $\mathbf{B}_{11}$ must be linearly independent. This implies invertibility.

32. If it is desired to include a direct effect of $\mathbf{v}_t$ on $\mathbf{k}_{t+1}$, this can be accomplished by defining an auxiliary variable (equal to $\mathbf{v}_{t-1}$) in $\mathbf{x}_t$ (in which case $\mathbf{v}_t$ remains in the information set for period *t*). Also, auxiliary variables can be used to include expectations of future values of exogenous variables.

33. Binder and Pesaran (1995) show that virtually any linear model can be put in form (1), but in doing so admit a more general specification than (2) for the process generating the exogenous variables.

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