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Saki Bigio
Eduardo Zilberman

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Agustinas 1180, Santiago, Chile
Teléfono: (56-2) 3882475; Fax: (56-2) 3882231

Speculation-Driven Business Cycles*

Saki Bigio
UCLA

Eduardo Zilberman
PUC-Rio
Central Bank of Chile

Abstract

Speculation, in the spirit of Harrison and Kreps [1978], is introduced into a standard real business cycle model. Investors (speculators) hold heterogeneous beliefs about firm growth. Firm ownership, and thus, the firm's discount factor varies with waves of optimism and leverage. These waves ripple into firm investments in hours. The firm's discount factor links the equity premium and labor volatility puzzles. We obtain an upper bound to the amplification that can be generated by speculation for any model of beliefs – a factor of 1.5. A calibration based on diagnostic beliefs amplifies hours volatility by a factor of 1.15 and produces a bubble component of 20 percent.

Resumen

Se introduce especulación, al estilo de Harrison y Kreps (1978), a un modelo de ciclos económicos reales: grupos de inversionistas (especuladores) difieren acerca de sus proyecciones de crecimiento de las firmas. La propiedad de las empresas varía conforme al grupo de inversionistas que deciden invertir en acciones. El optimismo y apalancamiento de aquellos inversionistas determina conjuntamente el factor de descuento que utiliza la firma para las decisiones de empleo. Este mecanismo vincula el exceso de retornos bursátiles a la volatilidad del empleo. Se obtiene una cota superior para la amplificación que pueden generar las diferencias de proyecciones económicas entre inversionistas: el ciclo económico se puede amplificar hasta un 50 por ciento. Para un ejercicio de calibración, la volatilidad del empleo se amplifica en un 15 por ciento y se produce componente burbuja de 20 por ciento.

*Bigio: sbigio@ucla.edu. Zilberman (corresponding author): zilberman@econ.puc-rio.br. We are grateful for helpful conversations with Sebastian Di Tella, Fatih Guvenen, Fernando Mendo and Alp Simsek. We also thank participants at internal presentations in the Central Bank of Chile and PUC-Rio, as well as at the Santiago Macro Workshop 2019 for helpful comments. The views expressed in this paper do not necessarily reflect the position of the Central Bank of Chile. All errors are ours.

“I’m happy to say I am a Harrison-Kreps-Keynesian.”

Thomas Sargent, New York Times, October 29, 2011.

1 Introduction

One of the most prominent views about deep economic downturns is that, at least in part, these are due to waves of speculation. This tradition can be traced to historical narratives found in the works of Hyman Minsky, Charles Kindleberger, and more recently of Robert Shiller. These narratives share common elements. A wave of excess optimism and disagreement about earnings-growth prospects leads to speculation. Speculation leads to asset bubbles, which are fueled by increased leverage. When bubbles burst in Wall Street, it echos in Main Street, the story goes. These narratives are supported by evidence on large stock market turnover, high price-earnings ratio (P/E) ratios, and expansionary measures in credit, which apparently increase the hazard rate of deep recessions [Shiller, 2000]. We can dub this view “speculation-driven business cycles.”

Theory has formally explained bubbles and asset-price crashes caused by waves of optimism and pessimism in ways that subtly depart from the discipline of rational expectations. This tradition can be traced back to Harrison and Kreps [1978], and more recently to Scheinkman and Xiong [2003]. These studies show that speculators that alternate in their degree of optimism, once limited by short-selling constraints, provoke asset price bubbles. In this literature, bubbles are speculative, because they follow from the idea that the asset can be resold to someone more optimistic about the future. A number of studies have followed this lead and we have good ideas about the model ingredients necessary to generate speculative bubbles [Xiong, 2013].

To the best of our knowledge, theories of speculative bubbles have not yet been incorporated into a standard business cycle models.¹ In fact, when questioned about rational expectations, Thomas Sargent gave the following example:²

¹There is a tradition regarding rational bubbles.

²*Interview with Thomas Sargent*, The Region, August 26, 2010. Available at <https://www.minneapolisfed.org/article/2010/interview-with-thomas-sargent>.

“[...] economists have been working hard to refine rational expectations theory. [...] An influential example of such work is the 1978 QJE paper by Harrison and Kreps. [...], for policymakers to know whether and how they can moderate bubbles, we need to have well-confirmed quantitative versions of such models up and running.”

Once one wants to take on Sargent’s challenge, it is not clear how to bring speculation into business cycle theory. This paper fleshes out a recipe to take on that challenge. It concludes with a quantitative exploration. We see this quantitative exploration as a first step toward having “well-confirmed quantitative versions” of speculation-driven business cycle models. This task is important. Policy should not care about speculation and bubbles *per se*, but only inasmuch as it is a source of generalized downturns.

We modify a standard RBC model so that it can speak to the narrative of speculation-driven business cycles. Aside the representative worker and representative firm, we introduce a set of investors who agree to disagree about the evolution of the growth rate of total factor productivity (TFP). We introduce several ingredients. The first ingredients generate business cycle fluctuations that stem from investor sentiments. These ingredients are time-to-build and belief heterogeneity. These features provoke fluctuations, even in absence of speculative bubbles. To fit the narratives, we need additional ingredients that generate speculative bubbles, and bubbles that manifests in high P/E ratios. These ingredients are short-selling constraints, alternating degree of optimism and asset-market segmentation.

We introduce time-to-build as in the original Kydland and Prescott [1982] framework, which translates into a model where labor is hired a period in advance. Thus, labor is a form of firm investment. This assumption is important to link asset prices with economic fluctuations. We present a formula connecting the firm’s discount across states to the excess returns in the economy. When excess returns are high, the firm’s discount factor is high. The connection between asset prices and business cycles emerges because the firm’s discount factor determines the investment in labor. This mechanism is grounded on evidence documented by Lustig and Verdelhan [2012] which finds that excess returns (adjusted for volatility) are higher in recessions than in expansions. As the authors emphasize,

due to this higher risk-adjusted costs of capital during recessions, even unconstrained firms should invest and hire less.³

To fit the narrative, excess returns must be linked to investor beliefs. Here, investors make portfolio decisions, borrowing and lending at a risk-free rate and investing in shares.⁴ When more optimistic investors increase their share ownership, they become more representative of the shareholder pool. Their wealth, leverage, and relative optimism, influences the firm's hiring decisions by determining the firm's discount factor. In the environment, the equity premium and labor volatility puzzles are intimately linked. We argue that for any configuration of belief heterogeneity, the counter-cyclical of risk premiums is amplified.

Time-to-build and heterogeneous beliefs are enough to amplify fluctuations. However, these features are not enough to produce a pattern that fits the narrative of speculation-driven business cycles. In principle, it is not obvious that excess returns are driven by stock returns, and not by fluctuations in the risk-free rate. For that, we need segmentation.⁵ Our formulation of segmented markets is motivated by Guvenen [2009]. Namely, he assumes segmented markets such that some agents (the worker in our case) do not participate in the stock market.⁶ With segmented markets, the price pressure of speculation is released through the stock price, and not through the risk-free rate, provided that the worker's net supply of funds is sufficiently interest-rate elastic.

A final aspect of the theory is that, as highlighted by Harrison and Kreps [1978], to generate speculation and asset-price bubbles, speculators must be subject to short-selling constraints, and alternate between optimism and pessimism.

³Recent work by Hall [2017] puts a similar mechanism to work in a Diamond-Mortensen-Pissarides framework. When discounts (or risk premiums) are high, firms invest less in creating jobs and, thus, unemployment increases. By imposing fluctuations in premiums that match the data, he shows the model can account for the bulk of fluctuations in unemployment.

⁴Since Mehra and Prescott [1985], the asset-pricing literature has found environments that can help explain the equity premium puzzle [Cochrane, 2017]. In particular, Bansal and Yaron [2004] show that to produce volatile stochastic discount factors we need recursive preferences, long-run risk, and time-varying volatility. We follow Epstein and Zin [1991] and Tallarini [2000], and endow speculators with non-separable recursive preferences and allow for "long-run" risks, shocks to firm TFP growth – rather than the level. Speculators disagree about the distribution of TFP growth rates. We let heterogeneous beliefs be the source of time-varying volatility.

⁵We demonstrate this through a simple formula for the stock price, which holds for any specification of beliefs and degrees of risk-aversion – and that holds for intertemporal elasticity of substitution approximately equal to one.

⁶Guvenen [2009] shows that this assumption, coupled with heterogeneous intertemporal elasticity of substitution among participants and non-participants, renders the model consistent with several features of asset prices, including high equity premium.

Hence, we need a parsimonious model of beliefs that not only generates this alternating pattern, but also can be disciplined by data. We obtain that by exploiting the work of Gennaioli and Shleifer [2010]. We assume that speculators are either rational (who hold the correct beliefs) or “diagnostic”. Gennaioli, Shleifer and co-authors argue that diagnostic expectations explain a myriad of social phenomena.⁷ Here, when shocks are persistent, diagnostic beliefs produce extrapolative behavior. Diagnostic investors expect shocks to be more persistent than they really are. As a result we get the alternating pattern. Diagnostics are over-optimistic in high growth states, but over-pessimistic in adverse states. Also, diagnostic beliefs are a convenient formulation for quantitative purposes, as these can be summarized with a single parameter.

With these features the framework fits the qualitatively pattern of the speculation-driven business cycle narrative. Waves of optimism amplify the business cycle as the optimistic investors tend to lever and buy shares. As good states persist, they accumulate wealth. This increasing wave of overoptimism leads to more willingness to bear risk, which induces lower excess returns. Since firm’s beliefs are adjusted for its stockholders’ stochastic discount factors, the firm employs more. The opposite happens upon a persistent negative shock. As pessimistic investors accumulate wealth, the economy displays less willingness to tolerate risk and high excess returns, which translate into less hours. The presence of diagnostic investors implies deeper recessions and large turnover after transitions from good to bad states. Indeed, diagnostics accumulate wealth during booms. Once a recession hits, they become pessimists but they remain relatively wealthy, influencing the overall mood and willingness to bear risk in the economy. This line of reasoning also implies asymmetric real business cycles, in the sense that the longer the boom persists, the more severe is the bust.

In comparison to an economy without short-selling constraints, the presence of these constraints pushes employment up in all states, whether recessive or expansionary. Intuitively, overall willingness to bear risk by the marginal buyer increases in all states due to the option value to resell firm shares after transitions. In addition, short-selling constraints make wealth a slower moving variable. Hence, the propagation of shocks tends to

⁷Related to this paper, Bordalo et al. [2018] and Bordalo et al. [forthcoming] show that it can explain how agents forecast stock returns, and Bordalo et al. [2018] argue that it can generate credit cycles.

last longer.

From a quantitative perspective, a standard calibration of the model generates stock market bubbles of the order of 20 percent relative to their fundamental value. The volatility of labor hours fluctuations can be amplified by a factor of 1.15 due to speculative behavior. We show that in our quantitative exploration, cycles can be amplified at most by a factor of 1.50. Hence we see this quantitative result as a sizable direct effect due to speculation, that would be amplified were other frictions, such as sticky prices or fire-sales externalities, accounted for.

2 Literature review

This paper is, of course, related to the large literatures on real business cycles and the equity premium puzzle originated in Kydland and Prescott [1982] and Mehra and Prescott [1985], respectively. As in our paper, a recent strand links fluctuations in risk premiums to real business cycles. Di Tella and Hall [2019] stresses the role of uninsurable idiosyncratic risk and precautionary savings, whereas Hall [2017], Borovička and Borovičková [2019] and Kehoe et al. [2019] study unemployment fluctuations in the context of the Diamond-Mortensen-Pissarides search model. Our paper also fits into the recent macro-finance literature that emphasizes the importance of the wealth share of special individuals (e.g., financial intermediaries) for the business cycle. For example, He and Krishnamurthy [2011], Brunnermeier and Sannikov [2014], Mendo [2018], among others. In our case, the wealth share of diagnostic investors is key.

In addition, our paper is also related to the natural selection literature, which asks whether those agents with incorrect beliefs eventually disappear. Blume and Easley [1992, 2006] and Sandroni [2000] argue that only those with more accurate beliefs survive in the long-run in an environment with complete markets and separable preferences. However, this result is not robust to the market structure, as shown by Beker and Chattopadhyay [2010], Blume et al. [2018] and Cao [2018], and also not robust to preferences that are non-separable recursive even when markets are complete, as shown recently by Dindo [2019] and Borovička

[forthcoming]. Closely related is Cao [2018], who works out the same investor's problem as ours, but does not link beliefs to TFP shocks in a RBC economy. In fact, the paper studies the natural selection hypothesis in an endowment economy with incomplete markets.

Below we confirm the natural selection hypothesis in an example with separable preferences and without short-selling constraints. In addition, despite the aforementioned recent contributions, in all simulations reported in the paper, rational investors eventually accumulate the entire stock of investors' wealth.

Regarding the literature on heterogeneous beliefs and speculative behavior, we borrow the key ingredients from Harrison and Kreps [1978] and Scheinkman and Xiong [2003]. The interaction with financial markets is explored by Geanakoplos [2003, 2010], Fostel and Geanakoplos [2008], Simsek [2013], Iachan et al. [2018], among others.

Other papers have studied the transmission of speculative behavior and bubbles to the real sector. In Gilchrist et al. [2005], monopolistic firms can overcome short-selling by issuing shares at a price above fundamental value, which lowers the cost of capital and enhance investment. Bolton et al. [2006] present an agency model in which over-investment occurs during a bubble episode due to stock-based executive optimal compensation contracts that emphasize short-term stock performance. In contrast, Panageas [2005] shows that once investment subject to quadratic costs is introduced in a model with heterogeneous beliefs and short-selling constraint, despite the speculative behavior of agents, the neoclassical q theory of investment remains valid. Related to our work, Buss et al. [2016] study policy implications in a quantitative framework in which agents trade for both risk-sharing and speculative reasons, and speculation reduces investment and welfare as it pushes the cost of capital up. Recently, Caballero and Simsek [2019] show how speculation between optimistic and pessimistic investors, by affecting the evolution of the distribution of wealth among them, amplify a recession generated by a decline in risky asset valuations when output is determined by aggregate demand.

By assuming some agents hold diagnostic beliefs in the spirit of Gennaioli and Shleifer [2010], this paper is also related to the literature that explores how subjective beliefs affects the business cycle. See, for example, Eusepi and Preston [2011], Angeletos et al. [2018], Bor-

dalo et al. [2018], Bhandari et al. [2019], among others. Relatedly, Adam and Merkel [2019] show that (homogeneous) extrapolative beliefs can explain the stock price and business cycles altogether. Both cycles are connected as high stock prices signal profitable investment opportunities to capital producers.

Finally, a large literature studies other types of bubbles that emerge for reasons other than heterogeneous beliefs, such as the so-called “rational bubbles” [Blanchard and Watson, 1982, Santos and Woodford, 1997]. Recent contributions emphasize the interaction of rational bubbles and policy, for example, Galí [2014], Hirano et al. [2015], Allen et al. [2018] and Asriyan et al. [2019]. We leave the study of the role of policy in versions of our speculation-driven business cycle framework for future research.

3 Environment

Consider an infinite-horizon closed economy set in discrete time ($t = 0, 1, \dots$). We introduce investors into a standard real business cycle (RBC) model with a representative worker and a representative firm. Investors (or potential speculators) differ in beliefs regarding the evolution of the growth of total factor productivity (TFP), and may hold (or issue) risk-free bonds and hold (or short-sell) risky shares of the firm. Workers do not hold stocks. The differences of beliefs induce the desire to lever and may introduce speculative portfolios, in the spirit of Harrison and Kreps [1978] and Scheinkman and Xiong [2003]. In addition, we assume the firm hires labor one period in advance. This links portfolio decisions and labor fluctuations, through the valuation of the firm.

3.1 Investors

The economy is populated by a finite number of infinite-lived investors, indexed by $i \in \{1, \dots, I\}$, and with corresponding masses μ_i . Investor i derives utility from the flow of consumption $c_{i,t}$. In particular, we adopt Epstein-Zin recursive preferences:

$$U_{i,t} = c_{i,t}^{1-\beta} \left(\mathbb{E}_{i,t} \left[U_{i,t+1}^{1-\gamma} \right] \right)^{\frac{\beta}{1-\gamma}}, \quad (1)$$

where $\beta \in (0, 1)$ is the discount factor and $\gamma \geq 0$ is the risk-aversion parameter. The coefficient associated with the intertemporal elasticity of substitution (IES) in the Epstein-Zin formulation is set to 1, so to obtain analytic expressions.⁸

Heterogeneity regards beliefs. Thus, expectations about future states, $\mathbb{E}_{i,t}$, are indexed by the agent identity. In particular, investor i forms beliefs $\{p_{ss'}^i\}$ regarding the TFP growth, g_t , which takes value in $\{\bar{g}_1, \dots, \bar{g}_S\}$. TFP growth may transit from state s to s' , and is assumed to follow a Markov process with S states. Differences in beliefs regarding TFP growth of the representative firm translate into differences in beliefs about its future profits and stock returns, which creates a motive for trade in the financial market.

Investor i chooses consumption $c_{i,t}$, shares of the representative firm $n_{i,t+1}$ and risk-free bonds $b_{i,t+1}$ to maximize (1) subject to the borrowing constraint,

$$c_{i,t} + q_t n_{i,t+1} + b_{i,t+1} = (q_t + \pi_t) n_{i,t} + R_t b_{i,t}.$$

Here, q_t is the price per share (the total amount of shares are normalized to 1), and π_t are the profits of the representative firm. Investors can increase their leverage by issuing bonds. To get speculative behavior, we study versions of the model that differ in the extent of short-selling constraints. But in general, both $b_{i,t+1}$ and $n_{i,t+1}$ can take negative values. Finally, R_t is the risk-free rate that accrues to bonds bought in period $t - 1$ and carried over period t .

Within this framework, investors who are overly optimistic about future TFP prospects will buy more shares and tend to issue bonds. The more pessimistic tend to save by holding bonds that yield the risk-free rate. As we explain below, differences in beliefs determine ownership, which in turn, defines the representative investor of the firm. This determines the firm's discount factor, which in turn, produces labor-market fluctuations.

⁸To obtain these preferences, just take the limit $\rho \rightarrow 1$ of the more standard formula in Epstein and Zin [1991],

$$U_{i,t} = \left\{ (1 - \beta) c_{i,t}^{1-1/\rho} + \beta \mathbb{E}_{i,t} \left([U_{i,t+1}^{1-\gamma}]^{\frac{1-1/\rho}{1-\gamma}} \right)^{\frac{1}{1-1/\rho}} \right\},$$

where ρ is the IES were the model deterministic.

3.2 Representative worker

The representative worker derives utility from consumption $c_{w,t}$ but disutility from labor hours h_t . We assume preferences are GHH. In particular, the worker chooses $c_{w,t} \geq 0$ and $h_t \geq 0$ to maximize

$$\mathbb{E}_{w,0} \left[\sum_{t=0}^{\infty} \beta^t u \left(c_{w,t} - \xi A_{t-1} \frac{h_t^{1+\nu}}{1+\nu} \right) \right], \quad \text{subject to} \quad c_{w,t} + B_{t+1} = w_t h_t + R_t B_t,$$

where $u' > 0$ and $u'' \leq 0$. $\xi > 0$ is the scale factor on labor disutility, $\nu > 0$ is the inverse of the Frisch elasticity, and $\beta \in (0, 1)$ is the discount factor. Finally, w_t is the wage rate, and B_{t+1} is the worker's savings.

The worker's budget constraint features an implicit assumption. Asset markets are segmented. In particular, the representative worker cannot hold shares.⁹ This assumption implies that the worker's savings B_{t+1} is a variable that determines an important feature of the model. This variable, as we explain below, determines how the impact of speculation is absorbed by the cost of capital. In particular, it determines whether speculation shows up in q_t or R_{t+1} . To make the arguments as simple as possible, we do not take a stance on how the worker makes savings-consumption decisions. In principle, next-period savings (or the supply of funds schedule) can be any function of the state variables, to be introduced in the next section, as long as it follows a balanced growth path. Hence, although our numerical simulations assume rule-of-thumb behavior, all analytical results derived in what follows are consistent with optimal behavior.

The disutility of labor supply also merits some discussion. The scaling factor A_{t-1} is necessary to guarantee the existence of a balanced growth path as in Jaimovich and Rebelo [2009]. It can be interpreted as a long-run wealth effect. Also, note that since the firm hires labor one period in advance, then h_t is the labor hired at $t-1$ but only supplied at t , when the worker experiences the disutility of working. In principle, the worker's perceived stochastic process of TFP growth, $\{p_{ss'}^w\}$, used to form expectations $\mathbb{E}_{w,t}$, can be different from the true process and from those perceived by the other agents in the economy.

⁹We could relax this assumption by assuming a less extreme form of market segmentation. The worker, for instance, could hold some shares, as long as some costs prevent the adjustment of the portfolio immediately.

3.3 Representative firm

The representative competitive firm hires labor h_{t+1} that will be employed at $t+1$ one period in advance (i.e., before the realization of the shock) to maximize expected profits,

$$\mathbb{E}_{f,t} [A_t g_{t+1} h_{t+1}^\alpha - w_{t+1} h_{t+1}],$$

where w_{t+1} is the wage rate paid at $t+1$. Given an initial level A_0 , TFP evolves according to $A_t = A_{t-1} g_t$, where $g_t \in \{\bar{g}_1 < \bar{g}_2 < \dots < \bar{g}_S\}$ follows a S -state Markov process with transition probabilities $\{p_{ss'}\}$. The current state s and future state s' take values in $\{1, \dots, S\}$. Qualitative results do not change if we assume disagreement regarding the evolution of the TFP level rather than its trend, but assuming the latter improves the behavior of risk premiums [Bansal and Yaron, 2004]. We assume the firm uses its own beliefs $\{p_{t,ss'}^f\}$ regarding the evolution of TFP growth to form expectations $\mathbb{E}_{f,t}$. Firm's beliefs reflect ownership. Given that labor is chosen one period in advance, probabilities are adjusted to account for the stockholders' stochastic discount factors. Akin to Hall [2017], fluctuations in risk premiums generate fluctuations in investment in hours. We spell out the precise formula for firm's beliefs below, after we introduce the recursive version of the model.

3.4 Market clearing

Given the same notation for labor h_t in both firm's and worker's optimization problems, we already impose market clearing in the labor market. By taking the first order conditions (FOCs) with respect to h_t in both problems, and equalizing supply and demand in the labor market,¹⁰ one obtains,

$$h_{t+1} = \left(\frac{\alpha \mathbb{E}_{f,t}[g_{t+1}]}{\xi} \right)^{\frac{1}{1+\nu-\alpha}} \quad \text{and} \quad \frac{w_{t+1}}{A_t} = \xi \left(\frac{\alpha \mathbb{E}_{f,t}[g_{t+1}]}{\xi} \right)^{\frac{\nu}{1+\nu-\alpha}}. \quad (2)$$

¹⁰The labor supply schedule and the labor demand schedule are $w_{t+1} = \xi A_t h_{t+1}^\nu$ and $w_{t+1} = \alpha A_t \mathbb{E}_{f,t}[g_{t+1}] / h_{t+1}^{1-\alpha}$, respectively.

Hence, realized profits at $t + 1$ are given by

$$\frac{\pi_{t+1}}{A_t} = \left(\frac{1}{\xi}\right)^{\frac{\alpha}{1+\nu-\alpha}} (\mathbb{E}_{f,t}[g_{t+1}])^{\frac{1+\nu}{1+\nu-\alpha}} \left\{ \frac{g_{t+1}}{\mathbb{E}_{f,t}[g_{t+1}]} \alpha^{\frac{\alpha}{1+\nu-\alpha}} - \alpha^{\frac{1+\nu}{1+\nu-\alpha}} \right\}.$$

To close the model, we specify the remaining market clearing conditions. Market clearing for goods requires that

$$\sum_{i=1}^I \mu_i c_{i,t} + c_{w,t} = A_{t-1} g_t h_t^\alpha,$$

whereas market clearing for bonds and shares require that

$$B_{t+1} = - \sum_{i=1}^I \mu_i b_{i,t+1} \quad \text{and} \quad \sum_{i=1}^I \mu_i n_{i,t+1} = 1,$$

respectively. The definition of the equilibrium is standard. The model features balanced growth path, with all variables growing at the same rate except labor n_t , risk-free interest rate R_{t+1} and shares holding $n_{i,t+1}$. Let $\hat{x}_t = x_t/A_{t-1}$ denote a generic de-trended variable.

4 Recursive formulation

The model features one exogenous state variable $s \in \{1, \dots, S\}$, which indexes the growth in TFP. Denote the current aggregate endogenous state variables by X , to be defined below, whereas the future ones by X' . Let the law of motion of X be given by a transition function ψ , such that $X' = \psi(X, s, s')$. Finally, the recursive formulation for firm's de-trended profits is the following,

$$\hat{\pi}(X, s, s') = \frac{1}{\xi^{\frac{\alpha}{1+\nu-\alpha}}} \left\{ \bar{g}_{s'} \left(\alpha \mathbb{E}_{X,s}^f[g] \right)^{\frac{\alpha}{1+\nu-\alpha}} - \left(\alpha \mathbb{E}_{X,s}^f[g] \right)^{\frac{1+\nu}{1+\nu-\alpha}} \right\}, \quad (3)$$

which depends on X only indirectly through firm's beliefs, that reflect firm's ownership as well as shareholders' stochastic discount factors, to be specified below.

The next subsections present the recursive representation of the model and some additional results.

4.1 Investor problem

Recall that de-trended variables are expressed as $\hat{x}_t = x_t/A_{t-1}$. To solve the investors' problem, we perform a change of variables. First, define the individual state to be

$$\hat{a}_{i,t} = (\hat{q}_t + \hat{\pi}_t)n_{i,t} + R_t \hat{b}_{i,t},$$

which is the investor i 's de-trended wealth in the current period. Hence, the budget constraint can be rewritten as

$$\hat{c}_{i,t} + \hat{q}_t n_{i,t+1} + g_t \hat{b}_{i,t+1} = \hat{a}_{i,t}.$$

Second, after omitting the subscript t , define $\tilde{c}_i = \hat{c}_i/\hat{a}_i = c_i/a_i$ as the consumption share of wealth; $\tilde{n}_i = \hat{q}n'_i/\hat{a}_i(1 - \tilde{c}_i)$ as the share of invested wealth, i.e. after deducing consumption, that is invested in risky shares; and, analogously, $\tilde{b}_i = g\hat{b}'_i/\hat{a}_i(1 - \tilde{c}_i)$ is the share of invested wealth that goes to risk-free bonds. Hence, the budget constraint can be written as $\tilde{n}_i + \tilde{b}_i = 1$.

Let the returns on shares be denoted by

$$R_n(X, s, s') = \left[\frac{\hat{q}(\psi(X, s, s'), s') + \hat{\pi}(X, s, s')}{\hat{q}(X, s)} \right] \bar{g}_s,$$

and the risk-free rate on bonds bought today and carried out until tomorrow be $R_b(X, s)$.

Next period de-trended wealth, after substituting $\tilde{n}_i = 1 - \tilde{b}_i$, is given by

$$\hat{a}'_i = \frac{1}{\bar{g}_s} \left[R_n(X, s, s')(1 - \tilde{b}_i) + R_b(X, s)\tilde{b}_i \right] \hat{a}_i(1 - \tilde{c}_i). \quad (4)$$

The optimization problem of investors can be written as

$$U_i(\hat{a}_i, X, s; A_{-1}) = \max_{\tilde{c}_i, \tilde{b}_i} \left\{ (\tilde{c}_i \hat{a}_i A_{-1})^{1-\beta} (\mathbb{E}_{i,s} [U_i(\hat{a}'_i, X', s'; A)^{1-\gamma}])^{\frac{\beta}{1-\gamma}} \right\},$$

subject to (4). For now, we consider the version without short-selling constraints. We follow a guess-and-verify method to characterize the solution. Conjecture that $U_i(\hat{a}_i, X, s; A_{-1}) = V_i(X, s)\hat{a}_i A_{-1}$. We prove the following lemma.

Lemma 1. *Given the law of motion $X' = \psi(X, s, s')$ and prices $R_b(X, s)$ and $R_n(X, s, s')$ for all X, s, s' , the optimal consumption share for investor i is $\tilde{c}_i(X, s) = 1 - \beta$, and the optimal portfolio weight $\tilde{b}_i(X, s)$ is defined implicitly by*

$$\mathbb{E}_{i,s} \left[\frac{(V_i(X', s'))^{1-\gamma} [R_b(X, s) - R_n(X, s, s')]}{\left[R_n(X, s, s')(1 - \tilde{b}_i(X, s)) + R_b(X, s)\tilde{b}_i(X, s) \right]^\gamma} \right] = 0. \quad (5)$$

In addition, V_i satisfy the following recursion,

$$\begin{aligned} \ln V_i(X, s) &= (1 - \beta) \ln \tilde{c}_i(X, s) + \beta \ln(1 - \tilde{c}_i(X, s)) + \frac{\beta}{1 - \gamma} \times \\ &\times \ln \left(\mathbb{E}_{i,s} \left[\left(V_i(X', s') \left[R_n(X, s, s')(1 - \tilde{b}_i(X, s)) + R_b(X, s)\tilde{b}_i(X, s) \right] \right)^{1-\gamma} \right] \right). \end{aligned}$$

Proof. By conjecturing that $U_i(\hat{a}_i, X, s; A_{-1}) = V_i(X, s)\hat{a}_i A_{-1}$, after a monotone transformation by taking logs, one obtains,

$$\ln V_i(X, s) + \ln \hat{a}_i = \max_{\tilde{c}_i, \tilde{b}_i} \left\{ (1 - \beta)(\ln \tilde{c}_i + \ln \hat{a}_i) + \frac{\beta}{1 - \gamma} \ln \left(\mathbb{E}_{i,s} \left[(V_i(X', s')\hat{a}_i')^{1-\gamma} \right] \right) \right\}.$$

By plugging (4) into the equation above, after some algebra, one verifies that $\ln \hat{a}_i$ cancels on both sides, and obtains,

$$\begin{aligned} \ln V_i(X, s) &= \max_{\tilde{c}_i} \{ (1 - \beta) \ln \tilde{c}_i + \beta \ln(1 - \tilde{c}_i) \} + \\ &+ \frac{\beta}{1 - \gamma} \ln \left(\max_{\tilde{b}_i} \left\{ \mathbb{E}_{i,s} \left[\left(V_i(X', s') \left[R_n(X, s, s')(1 - \tilde{b}_i) + R_b(X, s)\tilde{b}_i \right] \right)^{1-\gamma} \right] \right\} \right), \end{aligned}$$

which separates consumption from portfolio decisions. In particular, the FOC with respect to consumption yields $\tilde{c}_i(X, s) = 1 - \beta$ for all i and (X, s) , and the FOC with respect the portfolio weight yields equation (5). \square

4.1.1 Firm beliefs

Given the characterization above of the investor problem, now we are in a position to define firm's beliefs, $\{p_{ss'}^f(X)\}$, which reflect ownership and stochastic discount factors.

To spare notation, let $v_i(X, s, s') = V_i(\psi(X, s, s'), s')$. An inspection of FOC (5) reveals that the stochastic discount factor of the speculator reads

$$SDF_i(X, s, s') = \frac{(v_i(X, s, s'))^{1-\gamma}}{\left[R_n(X, s, s')(1 - \tilde{b}_i(X, s)) + R_b(X, s)\tilde{b}_i(X, s) \right]^\gamma}. \quad (6)$$

Invoking equation (4) and the fact that $\hat{c}_i = (1 - \beta)\hat{a}_i$, the stochastic discount factor reads $v_{i,t+1}^{1-\gamma} \beta^\gamma \left(\frac{g_t \hat{c}_{i,t+1}}{\hat{c}_{i,t}} \right)^{-\gamma}$, a more familiar representation.

We let firm's beliefs be given by investors' risk-adjusted beliefs averaged across them. In particular,

$$p_{ss'}^f(X) = \sum_i \omega_i(X, s) \left[\frac{SDF_i(X, s, s') p_{ss'}^i}{\sum_{s'} SDF_i(X, s, s') p_{ss'}^i} \right], \quad (7)$$

where $\left\{ \frac{SDF_i(X, s, s') p_{ss'}^i}{\sum_{s'} SDF_i(X, s, s') p_{ss'}^i} \right\}_{s'}$ are the risk-neutral transition probabilities (or risk-adjusted beliefs) of investor i regarding all possible future states, and $\{\omega_i(X, s)\}_i$ are weights (summing one) that reflect ownership of the firm by investors.

Risk-neutral probabilities are simply beliefs adjusted for stochastic discount factors. The idea is that the firm maximizes on behalf of its shareholders, and internalizes that employment decisions induce profit risk due to the assumption that they must be taken one period in advance. Since the interaction of profit risk and portfolio decisions affects investors' well being, firm's beliefs must be adjusted in a way that maximizes shareholders' preferences. Risk-neutral probabilities accomplish this goal. Importantly, portfolio and employment decisions become closely connected, tying fluctuations in stochastic discount factors to fluctuations in labor hours.

Risk-adjusted beliefs across investors are aggregated using weights $\{\omega_i(X, s)\}_i$. For now, we do not take a stance on the weights given to investors. One possible example is to assume that weights are given by the proportions of shares. In this case,

$$\omega_i(X, s) = \frac{\mathbf{I}_{\{\tilde{n}_i(X, s) > 0\}} \eta_i \tilde{n}_i(X, s)}{\sum_i \mathbf{I}_{\{\tilde{n}_i(X, s) > 0\}} \eta_i \tilde{n}_i(X, s)},$$

where $\tilde{n}_i(X, s) = 1 - \tilde{b}_i(X, s)$ is the proportion of wealth after deducing consumption that is invested in risky shares, and $\eta_i = \frac{\mu_i \hat{a}_i}{\sum_i \mu_i \hat{a}_i}$ is the wealth share of speculator i .

4.1.2 Specialization: $S = 2$ and $\gamma = 1$

In Appendix A, we workout FOCs (5) and SDFs (6) when $S = 2$, say $s \in \{L, H\}$ with $\bar{g}_L < \bar{g}_H$. $S = 2$ is a natural benchmark. Absent any bounds on portfolio choices as assumed in this subsection, it renders market completeness due to the presence of two assets (risk-free bonds and risky shares) that can be used to transfer consumption through time and across states.

For $\gamma \neq 1$, the solution makes explicit the dependence of $\tilde{b}_i(X, s)$ and $SDF_i(X, s, s')$ on $v_i(X, s, s') = V_i(\psi(X, s, s'), s')$, for which we do not have close form solution. When $\gamma = 1$, this dependence on $v_i(X, s, s')$ is eliminated, which enhances tractability. The quasi-closed forms in the following Lemma are obtained.

Lemma 2. *Let $\gamma = 1$ and $S = 2$ with $s \in \{L, H\}$, then the optimal portfolio decision is*

$$\tilde{b}_i(X, s) = p_{sL}^i \frac{R_n(X, s, H)}{R_n(X, s, H) - R_b(X, s)} - p_{sH}^i \frac{R_n(X, s, L)}{R_b(X, s) - R_n(X, s, L)}.$$

In addition, individual wealth evolves according to

$$\frac{\bar{g}_s \hat{a}'_i(X, s, s')}{\beta \hat{a}_i} = \frac{1}{SDF_i(X, s, s')} = p_{ss'}^i R_b(X, s) \left[\frac{R_n(X, s, H) - R_n(X, s, L)}{|R_b(X, s) - R_n(X, s, -s')|} \right].$$

Hence, for any configuration of weights $\omega_i(X, s)$, firm's probabilities adjusted for the investors' stochastic discount factors are given by

$$p_{ss'}^f(X) = \frac{|R_b(X, s) - R(X, s, -s')|}{R_n(X, s, H) - R_n(X, s, L)} \quad (8)$$

Proof. Plug $\gamma = 1$ in equations (12) and (13) in Appendix A. □

When $\gamma = 1$ (log preferences) and $S = 2$, the model is highly tractable. Interestingly, by multiplying SDF_i and $p_{ss'}^i$ to compute shareholder i 's risk-adjusted beliefs in (7), $p_{ss'}^i$ in the numerator and in the denominator cancel each other. Hence, risk-adjusted beliefs are the same for all investors, being an explicit function of excess returns. In turn, excess returns also reflect the degree of belief heterogeneity in the economy.

This result has two implications. First, we do not need to take a stance on how adjusted beliefs are aggregated to form the firm's beliefs. As long as weights $\{\omega_i(X, s)\}$ sum to one, they can be anything.

Second, the simple formula in (8) connects two branches of modern macroeconomics. To the extent that firm's beliefs depend on excess returns, it ties the equity premium puzzle to the real business cycles, in particular labor market fluctuations. Intuitively, when excess returns are low (good state), and therefore investors are willing to bear risk, probabilities are distorted favoring the good state, and more labor hours are employed. Analogously, when excess returns are high, the firm employs less. In that sense, the inability of a standard real business models to generate both large fluctuations in hours (without relying on a large labor supply elasticity) and large volatility in the equity premium can be interpreted as a single puzzle.

4.1.3 Upper bound on \tilde{b}_i

Given that the FOCs with respect to the portfolio decisions hold with equality, subjective valuations of dividend flows of all speculators must coincide and, hence, must equal the price per share. As Miller [1977] and Harrison and Kreps [1978] emphasize, the presence of short-selling constraints generates subjective valuations (based on perceived dividend flows) that differ from the price per share. This leads to the possibility of speculative bubbles, something we explore in the paper.

Toward that end, we study what occurs when we impose a margin constraint on shares, $\tilde{b}_i \leq 1 + \kappa$. The coefficient $\kappa \geq 0$ is introduced to avoid excessive short-selling. In the extreme case of no short-selling, the constraint $n_{i,t+1} \geq 0$ must be satisfied, which is equivalent to $\tilde{b}_i \leq 1$. When $\kappa > 0$, then $q_t n_{i,t+1} \geq -\kappa \beta a_{i,t}$ is equivalent to $\tilde{b}_i \leq 1 + \kappa$, meaning that wealthier investors can short-sell proportionally more. If this upper bound is binding, $\tilde{b}_i(X, s) = 1 + \kappa$, the evolution of individual de-trended wealth in (4) reads

$$\hat{a}'_i(X, s, s') = \frac{\beta}{g_s} [(1 + \kappa)R_b(X, s) - \kappa R_n(X, s, s')] \hat{a}_i.$$

4.2 State variables

Now we are in a position to introduce the endogenous aggregate state variables that determine X . One natural candidate is the whole distribution of wealth: $R\hat{B}$ for the worker,¹¹ and $\{\hat{a}_i\}_{i=1}^I$ for speculators. We can encode the wealth distribution among speculators by keeping track of aggregate wealth $\sum_{i=1}^I \mu_i \hat{a}_i$ coupled with the wealth shares for $I - 1$ agents, i.e. $\{\eta_i\}_{i=1}^{I-1}$, where $\eta_i = \mu_i \hat{a}_i / \sum_{i=1}^I \mu_i \hat{a}_i$.

Nonetheless, although R is pre-determined, $\sum_{i=1}^I \mu_i \hat{a}_i$ depends on price \hat{q} which is an equilibrium object. To circumvent this problem, and get a pre-determined state variable, we use the definition of individual wealth, the market clearing conditions for bonds and shares, and the government budget constraint, to get

$$\sum_i \mu_i \hat{a}_i = \hat{q}(X, s) + \underbrace{\bar{g}_s h^\alpha - \hat{w}h}_{=\hat{E}} - R\hat{B}. \quad (9)$$

Note that \hat{E} and $R\hat{B}$ are pre-determined. Hence, $X \equiv \{R\hat{B}, \hat{E}, \{\eta_i\}_{i=1}^{I-1}\}$ is the set of aggregate endogenous state variables. Both \hat{E} and $R\hat{B}$ have simple interpretations. The former is de-trended realized profits (dividends received by shareholders), whereas the latter is the aggregate previous de-trended debt the speculators need to honor or the worker's current wealth.

4.3 Equilibrium

The next proposition derives an equilibrium relationship between the share price, current profits, and “net liquidity” available to speculators. Fix $\hat{B}'(X, s)$ which is the worker's de-trended aggregate savings carried out from today to tomorrow. We take it as a partial equilibrium object, albeit a useful one.

¹¹As opposed to R and B , which are the risk-free interest rate R on bonds B issued yesterday, we coupled the subscript b and the superscript prime to denote the risk-free interest rate $R_b(X, s)$ on bonds $B'(X, s)$ issued today, respectively. Finally, by de-trending the worker's budget constraint, one obtains $\hat{c}_{w,t} + g_t \hat{B}_{t+1} = \hat{w}_t h_t + R_t \hat{B}_t$.

Proposition 1. *The price per share is given by*

$$\hat{q}(X, s) = \frac{1}{1 - \beta} \left[\beta \hat{E} + \bar{g}_s \hat{B}'(X, s) - \beta R \hat{B} \right]. \quad (10)$$

Proof. We further develop equation (9) by using the market clearing condition for goods, the optimal consumption plan $\hat{c}_i = (1 - \beta)\hat{a}_i$ from the investors' optimization problem, the worker's budget constraint, and the equilibrium labor market outcomes in equation (2). Hence, the following equation is obtained,

$$\beta \sum_i \mu_i \hat{a}_i = \hat{q}(X, s) - \bar{g}_s \hat{B}'(X, s).$$

By plugging this expression back into equation (9), one obtains equation (10). \square

Equation (10) is a quasi-closed form for the price per share. It is derived using all equilibrium conditions in the model, except the FOCs (5) with respect the portfolio choices of the investors. This equation is fairly general, holding for any value of the risk aversion parameter γ , any configuration of beliefs $\{p_{ss'}^i\}$ including homogeneous beliefs, and any supply of funds schedule $\hat{B}(X, s)$ including the optimal one. It also holds if bounds, such as debt and/or short-selling constraints, are imposed on portfolio decisions. Hence, it is consistent with speculative behavior. Critical in the equation, however, is that the IES equal to one anchors the marginal propensity to consume out of wealth to $1 - \beta$. In more general formulations, the marginal propensity to consume is time varying, but remarkably stable – hence the equity premium puzzle. We lever on this result to argue that equation (10) captures the main quantitative forces that drive share price dynamics.

In particular, equation (10) directly links the price per share $\hat{q}(X, s)$ to discounted profits, $\beta \hat{E}$, and a measure of flow of funds (or “net” liquidity available) to investors, $\bar{g}_s \hat{B}'(X, s) - \beta R \hat{B}$. Both expressed in present value, given the $1 - \beta$ in the denominator.

If the worker is hand-to-mouth such that $\hat{B}'(X, s) = \hat{B} = 0$, then the P/E ratio, $\frac{\hat{q}(X, s)}{\hat{E}} = \frac{\beta}{1 - \beta}$, is constant. This is true even in the presence of the ingredients that make the economy prone to speculative bubbles. Hence, aside dynamics effects on asset prices through

the evolution of profits, $R_b(X, s)$ absorbs the effects of speculation. A bubble component, meaning that current asset price $q(X, s)$ is above all subjective valuations (based on the investors' perceived dividend flows), may arise because future dividends are discounted at a higher interest rate.

To get a pro-cyclical P/E ratio, the flow of funds to investors relative to current profits, $\frac{\bar{g}_s \hat{B}'(X, s) - \beta R \hat{B}}{\bar{E}}$, must be higher at the good states, which of course, depends on how $\hat{B}'(X, s)$ is specified, although there is a force through growth \bar{g}_s pushing funds up at good states. As a final extreme example, if liquidity supply is unlimited at a given interest rate R , speculation only affects $\hat{q}(X, s)$.

This discussion emphasizes the importance of market clearing conditions, as well as the net liquidity supply available to investors, provided by non-participating agents in the stock market. To sum up, given (X, s) , if “appetite” for shares (in fixed supply) due to speculation is high, the extent in which $\hat{q}(X, s)$ or $R_b(X, s)$ will reflect such “appetite” depends on the flow of funds to speculators.

In Appendix B, we describe the remaining equations that characterize the equilibrium, and outline an algorithm to solve the model numerically.

4.3.1 Bubble definition

In the presence of a bidding short-selling constraint, and alternating optimism and pessimism among speculators, the economy is prone to bubbles. Following Scheinkman and Xiong [2003], we define the bubble component encoded in the the price per share as the log-difference of the price per share and the largest subjective valuation,

$$\log \hat{q}(X, s) - \log(\max_i \hat{q}_i(X, s)),$$

where the subjective valuation $\hat{q}_i(X, s)$ satisfies the following recursion:

$$\hat{q}_i(X, s) = \sum_{s'} p_{ss'}^i SDF_i(X, s, s') (\hat{q}_i(\psi(X, s, s'), s') + \hat{\pi}(X, s, s')) \bar{g}_s.$$

In words, the bubble component is precisely the resale option value of the marginal buyer of shares.

4.3.2 Specialization: $S = 2$ and $\gamma = 1$

Recall that, absent any bounds on portfolio choices, $S = 2$ implies complete markets. In addition, $\gamma = 1$ makes the model highly tractable. If $S = 2$, $\gamma = 1$, and there are no short-selling constraints ($\kappa = \infty$), we get fairly simple formulas for portfolio shares and the evolution of wealth, as Lemma 2 highlights. In addition, we also get a simple expression for the evolution of wealth shares $\{\eta_i\}$, with a precise implication for its limit distribution.

Lemma 3. *If $\gamma = 1$, $S = 2$ with $s \in \{L, H\}$, and $\kappa = \infty$, then the evolution of the individual investor j 's wealth share is governed by*

$$\eta_j'(X, s, s') = \frac{p_{ss'}^j \eta_j}{\sum_{i=1}^I p_{ss'}^i \eta_i}, \quad \text{which implies} \quad \frac{\eta_j'(X, s, s')}{\eta_k'(X, s, s')} = \frac{p_{ss'}^j \eta_j}{p_{ss'}^k \eta_k} \text{ for all } j, k. \quad (11)$$

In addition, if one investor is rational, say speculator $i = 1$ such that $p_{ss'}^1 = p_{ss'}$ for all s, s' , then $\eta_1 \rightarrow 1$ almost surely.

Proof. The first part is a direct implication of Lemma 2. To prove the second part, note that there are I absorbing states, each with $\eta_j = 1$ for some j , and $\eta_i = 0$ for $i \neq j$. Indeed, suppose an absorbing state features $\eta_j \in (0, 1)$ for some j s. Hence, $\sum_{i=1}^I p_{ss'}^i \eta_i = p_{ss'}^j$ for all j such that $\eta_j \in (0, 1)$, which contradicts heterogeneous beliefs. With an abuse of notation, upon reaching state s today, denote the continuation history m periods ahead by $s^m|s = \{s, s_1, s_2, \dots, s_m\}$. Investor j attaches probability $\mathbf{p}^j(s^m|s) = p_{s, s_1}^j \prod_{t=1}^{m-1} p_{s_t, s_{t+1}}^j$ that such history will occur. In addition, given that investor $i = 1$ has the correct beliefs, $\frac{\mathbf{p}^j(s^m|s)}{\mathbf{p}^1(s^m|s)}$ is a non-negative martingale and, thus, converges almost surely. Hence, $\frac{\eta_j}{\eta_1}$ also converges almost surely, meaning that $\eta_1 \rightarrow 1$ almost surely. \square

Regarding the evolution of the wealth distribution, upon the realization of s' , the larger the ratio $p_{ss'}^j/p_{ss'}^k$, the higher the increase in relative wealth, η_j/η_k . Intuitively, investors who believed that the realized state s' was more likely to realize ex-ante, also chose portfolios that perform better in such state.

Regarding the limit result, only rational investors (i.e., those with correct beliefs) survive. In other words, they eventually acquire the whole investors' stock of wealth. This is a well-known result in a complete markets context with non-recursive preferences. See, for example, Sandroni [2000] and Blume and Easley [2006]. In the next section, we assume a rational investor exists. But if none of the investors are rational, one can extend this result by following the steps in Blume and Easley [2006], and show that investors with the closest beliefs to the truth survive.

Finally, the limit result is not general to other values (rather than one) of the risk aversion parameter γ , as shown by the recent contribution of Borovička [forthcoming]. It is also not readily generalizable within an incomplete markets structure in the presence of a debt-limit constraint, even when $\gamma = 1$, as Beker and Chattopadhyay [2010] and others show in related contexts.

4.4 Discussion

The model we have discussed so far is tailored to speak to many features of the speculation-driven business cycles. This can be easily seen for the specialization in which $\gamma = 1$ and $S = 2$, with $s \in \{L, H\}$. As the good state $s = H$ persists, the law of motion for wealth shares, equation (11), implies that optimistic investors accumulate wealth on average. Hence, asset prices reflect this increasing overoptimistic view. As in a standard macro-finance model, this wave of overoptimism leads to more willingness to bear risk, and this induces lower excess returns. The transmission from lower excess returns to higher hours worked can be seen through adjusted firm's beliefs in equation (8). A great willingness to bear risk not only affects excess returns, but also distort firm's beliefs in favor of good states. Whether lower excess returns reflect a higher risk-free interest rate or higher asset prices (i.e., lower asset returns) depends on the supply of liquidity to speculators. As equation (10) highlights, if high relative to profits during good times, the economy displays high P/E ratio, credit expansion and increased leverage, commonly associated to speculative episodes. The counterpart of this argument as the bad states persist also holds. As pessimistic investors accumulate wealth, the economy displays higher excess returns and lower labor hours.

By contrast to the most RBC models, technological shocks affect growth rates rather than levels. This is a stand in for growth prospects in the economy. Nonetheless, the same conclusions would emerge if we had assumed TFP shocks at the level in an economy without trend growth. Indeed, by reinterpreting g_t as levels, it is enough to substitute \bar{g}_s for one in all equations above. We opt to keep growth shocks for two reasons. First, they are important to make stochastic discount factors more volatile, and thus, generate somewhat larger risk premiums. Second, as equation (10) highlights, \bar{g}_s pushes the net flow of funds to speculators up at good states, something the model requires to generate pro-cyclical P/E ratio.

The next section takes a stand on the heterogeneity of beliefs, as a final step to make the model speaks comprehensively to the speculation-driven business cycles view.

5 Rational and diagnostic expectations

So far, we have been agnostic about beliefs, but in order to make progress toward quantitative statements, we need a model of beliefs. Hence, the last step to formalize the speculation-driven business cycle view is to impose some discipline on the heterogeneity of beliefs. We take a stance on the number of participants and assume $I = 2$ types of investors. We let $i = 1$ represent the rational investor who holds correct beliefs, $p_{ss'}^1 = p_{ss'}$ for all s, s' . We call $i = 2$ the “diagnostic” investor.

Diagnostic expectations are formalized by Gennaioli and Shleifer [2010, 2018] based on prior work by Daniel Kahneman and Amos Tversky, and applied by them and co-authors to explain a wide range of social phenomena. Suppose an agent wants to form beliefs regarding the distribution of types (future shock s' in our case) in a given group (current s in our case). Then a specific type (say s') is diagnostic or representative of this group if its true probability of realization ($p_{ss'}$) is large relative to its true probability of realization in some reference group (for example, all shocks other than s such that this probability is $\sum_{k \neq s} p_{ks'}$). Diagnostic expectations attribute more weights to diagnostic types.

We follow Bordalo et al. [2018], who formalize diagnostic expectations in the context of an AR(1) process. As reference group, they consider past conditions as if no news were

received in the meantime. If past conditions mean one period with no news, the probability of realization of s' in such reference group is $\sum_k p_{s_{-1}k} p_{ks'}$, which comes at a cost of tracking one more state variable, the previous shock s_{-1} . This extra state variable would be avoided if the reference group is described by conditions in the long past, such the probability of s' realizes is $\sum_k \bar{p}_k p_{ks'}$, where $\{\bar{p}_s\}$ is the invariant distribution associated with $\{p_{ss'}\}$. In particular, $i = 2$'s diagnostic beliefs are given by:

$$p_{ss'}^2 = p_{ss'} \left(\frac{p_{ss'}}{\sum_k \bar{p}_k p_{ks'}} \right)^\theta Z_s,$$

where Z_s is a constant that guarantees that $\sum_{s'} p_{ss'}^2 = 1$, and $\theta > 0$ measures the extent to which beliefs are distorted.

Hence, given that $\theta > 0$, beliefs are distorted by attributing more probability towards types (shocks) that are more diagnostic. Psychologically, diagnostic types are oversampled from limited and selective memory. Important for our purposes is the relevance of diagnostic expectations to explain how individuals forecast stock market returns [Gennaioli et al., 2016, Bordalo et al., forthcoming].

If $S = 2$ with $s \in \{L, H\}$, a simple algebra reveals that $p_{LL}^2 > p_{LL}$ and $p_{HH}^2 > p_{HH}$ if and only if $p_{LL} p_{HH} > (1 - p_{LL})(1 - p_{HH})$. In words, whenever shocks are persistent, diagnostic speculators believe that states are more persistent than they really are. This is the case for TFP shocks as observed in the data.

This simple and intuitive result has two implications for the speculation-driven business cycle view we evaluate in this paper. First, the diagnostic investor is optimistic at the good state but pessimistic at the bad state. Under the presence of a binding short-selling constraint, this observation allows the possibility of bubbles, as noted by Harrison and Kreps [1978]. Indeed, at the good state, the price of the share not only reflects the diagnostic's valuation but also the option to resell the share to the rational agent if the bad state realizes. Similarly, at the bad state, a bubble may arise as the optimistic rational is willing to pay a higher price than valuation due to the resell option. As salient during speculation episodes, with the presence of diagnostic investors and short-selling constraints, turnover of shares

is amplified, and a bubble component arises.

Second, the diagnostic agent accumulates proportionally more wealth as the states (whether good or bad) persist whereas the rational accumulates when the states transit. This observation implies that the longer is the boom within a cycle, the larger will be the drop in employment once the bust arrives. Indeed, as the good state persists, the optimistic diagnostic investor accumulates an increasing amount of wealth, also meaning an increasing amount of hours worked due to the aforementioned transmission mechanism. Once the bad state realizes, the diagnostic becomes pessimistic. The larger is the wealth previously accumulated, the larger is the fall in hours employed. Indeed, a wealthier pessimistic investor at the bad state implies that less hours are employed. We illustrate this and other implications of the model in the next session.

6 Speculation-driven business cycles

In this section we illustrate the mechanics of the model through some simulation exercises. We consider two cases. In a benchmark case, we assume the worker is hand-to-mouth and, thus, there is no flow of funds to speculators, $\hat{B}'(X, s) = \hat{B} = 0$. Then, we show how a proper supply of funds changes the financial moments implied by the model, but not the key transmission mechanism and the amplification of shocks. Appendix B describes the algorithm employed to solve the model numerically.

To highlight the role of speculation and bubbles, in both cases we present results with and without short-selling constraints. As discussed above, a short-selling constraint means that \tilde{b}_i is bounded above by $1 + \kappa$, where $\kappa = 0$ implies no short-selling at all. Except for Proposition 1, the analytical results obtained throughout the paper are not generalized when short-selling constraints bind for some speculators. Nonetheless, as discussed in Appendix B, we can easily implement these constraints numerically.

The calibration is purely illustrative. We set a symmetric Markov chain, with $p_{ss'} = 0.70$ if $s' = s$, and $\theta = 2$, such that $p_{ss'}^2 = 0.927$ if $s' = s$. Hence, there are lots of disagreement in this specification. In addition, $\beta = 0.99$, $\nu = 0.5$, $\xi = 1$, $\gamma = 1$, $\alpha = 2/3$, $\bar{g}_L = 1$ and $\bar{g}_H = 1.02$.

6.1 No flow of funds to investors, $\hat{B}'(X, s) = \hat{B} = 0$

In this section, we present simulations for the case in which the worker is hand-to-mouth, so there is no flow of funds to investors, $\hat{B}'(X, s) = \hat{B} = 0$. In Appendix B.1, we show that this assumption coupled with $\gamma = 1$, $I = 2$ and $S = 2$ simplifies the solution and the numerical implementation.

Figures 1 and 2 propose the following experiment. Suppose the economy is in steady-state with only rational investors, $\eta_1 = 1$, which is an absorbing state. After a sequence of good shocks, we simulate four consecutive periods of recession within twenty periods of expansion. The key distinction between the dashed and the filled lines is that the dashed one contemplates an “optimism” (or “diagnostic”) shock at $t = 1$, such that a mass of speculators holding twenty percent of the wealth becomes diagnostic. Investors are allowed to short-sell as they wish ($\kappa = \infty$) and, thus, bubbles cannot arise in this case. In contrast, the dotted line imposes no short selling at all ($\kappa = 0$), which generates a bubble. Figure 1 plots the evolution of the rational’s wealth share and (log) hours worked, whereas Figure 2 shows the evolution of the financial market outcomes (excess returns, firm’s adjusted probabilities, share’s price, risk-free rate and bubble component).

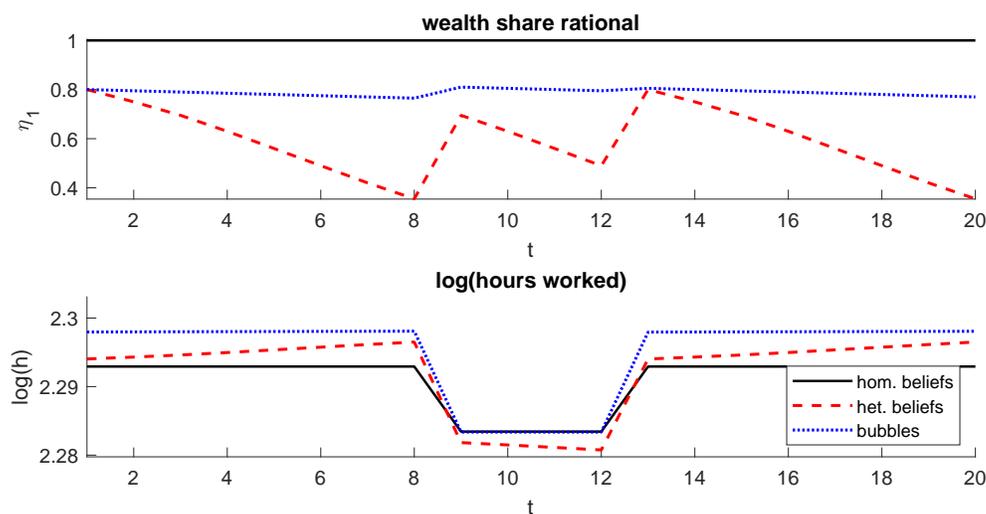


Figure 1: Real business cycle

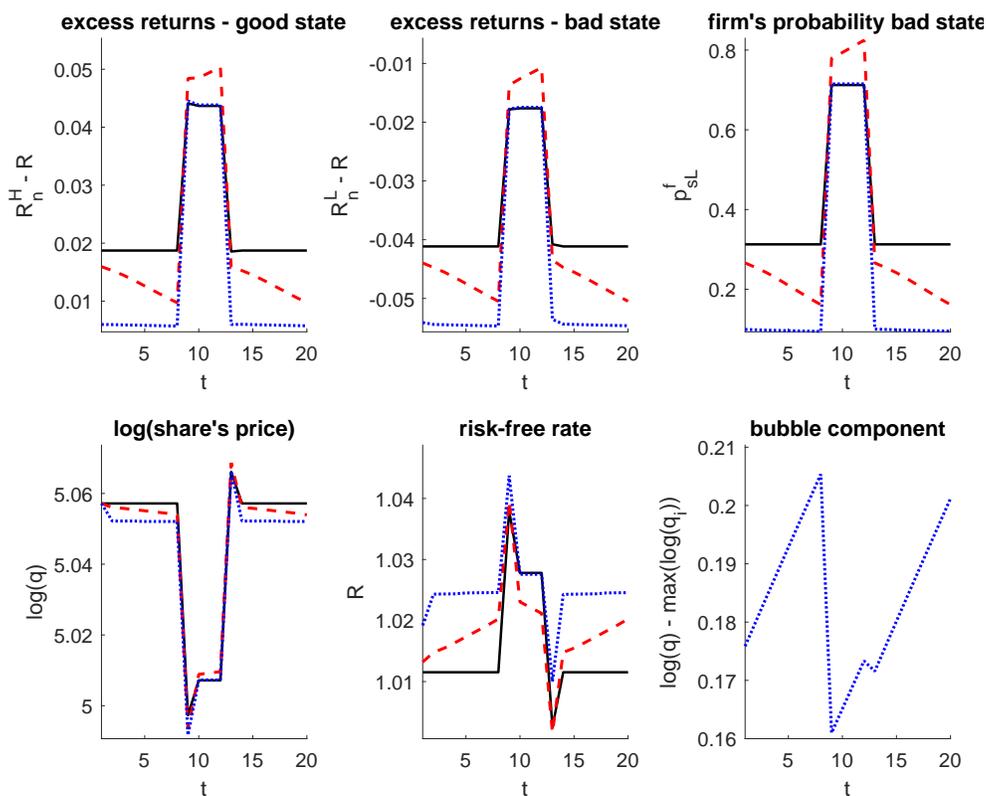


Figure 2: Financial market outcomes

Consider, first, the case without short-selling constraints ($\kappa = \infty$). The filled and dashed lines represent economies with homogeneous and heterogeneous beliefs, respectively. The economy with heterogeneous beliefs generates persistence and amplifies the cycle. As the good state $s = H$ persists, an increasing overoptimism dominates the economy. With a greater overall willingness to bear risk, excess returns decline as the boom persists. As a consequence, firm's adjusted probability of the occurrence of the bad state also declines and, thus, employment increases. Once the bust arrives, the diagnostic investors become pessimistic. Although their wealth is hit by the recession, they remain relatively wealthy. Hence, employment falls to a level below its counterpart in an economy with only rational investors. As the bad state persists, the diagnostic investors further accumulate wealth implying an increasing pessimistic view and, thus, increasing excess returns and declining labor hours.

Two conclusions emerge from Figures 1 and 2 once a short-selling constraint is imposed (dotted lines). First, the rational's wealth becomes a slower moving variable and fluctuate less. This implies that conditional on being in a good state, risk premiums, adjusted probabilities and hours employed are also slower moving variables. Second, with respect to the economy without short-selling constraints, the firm employs more hours at both the good and bad states. Intuitively, once the short-selling constraint is imposed, overall willingness to bear risk by the marginal buyer increases in both states, due to the bubble component that captures the option to resell after states transit. In particular, at the bad state when the rational speculator is the marginal buyer, outcomes are too close to the economy with only the rational investor. Similarly, at the good state, outcomes reflect more the optimistic beliefs of the diagnostic speculator, who is the marginal buyer. Hence, excess returns are pushed down in both states, meaning that firm's adjusted probabilities place more weight on good states.

As highlighted by equation (10), without supply of funds by non-participants in the stock market, general equilibrium forces imply a constant P/E ratio. In that case, excess returns movements are mostly due to the risk-free rate (as opposed to stock returns), as it adjusts more to motives to trade in the financial markets. As emphasized by Harrison and Kreps [1978] and Scheinkman and Xiong [2003], due to the resell option of shares upon a transition, the presence of short-selling constraints make the economy prone to bubbles in asset prices. Counter to the common view, however, as the risk-free rate is the price that mostly absorbs speculation, a bubble component arises not because the price per share increases, but because subjective valuations (of the perceived flow of dividends) fall due to higher discounting.

Some key financial variables that speak to the speculative-driven business cycle view are qualitatively aligned with the data. Indeed, excess returns decline at the good state and increase in the bad. Also, the model captures the boom-bust narrative of bubbles once the states transit. Nonetheless, some financial moments are at odds with the data. The price per share declines (increases) conditional on being in a good (bad) state. Also, the P/E ratio is constant (not reported), although they are pro-cyclical in the data. Finally, as

in a standard RBC model, we also find a counter-cyclical risk-free rate. But note that such counter-cyclicity is mitigated once heterogeneous beliefs and short-selling constraints are introduced.

Despite missing such moments, excess returns are the relevant financial moments for portfolio decisions, and the transmission mechanism from financial to real business cycles. The bubble component, independently if generated by high price per share or high interest rate, is also relevant to the extent that pushes excess returns up. Although contrafactual, abrupt movements in the price of the share and the risk-free rate kind of cancel each other to generate smooth excess returns conditional on being in a state. As we emphasize throughout the paper, the supply of funds to speculators is crucial to get the full picture of the financial markets right. In the next subsection, we work out an example that qualitatively accomplishes it. Nonetheless, as long as the supply of funds does not affect overall willingness to bear risk in the economy, it should not affect excess returns and business cycles according to the forces that hinge on the transmission mechanism proposed this paper.

Finally, Figure 3 reproduces two cycles with diagnostic shocks at $t = 1$ and $\kappa = \infty$. The dot-dashed line postpones the bust for four periods. The longer the boom persists, the more severe is the bust. This is an immediate implication of the reasoning above due to the extra four periods diagnostic investors accumulate wealth. Hence, the model delivers an explanation on why cycles are asymmetric despite we calibrate the Markov chain symmetrically.

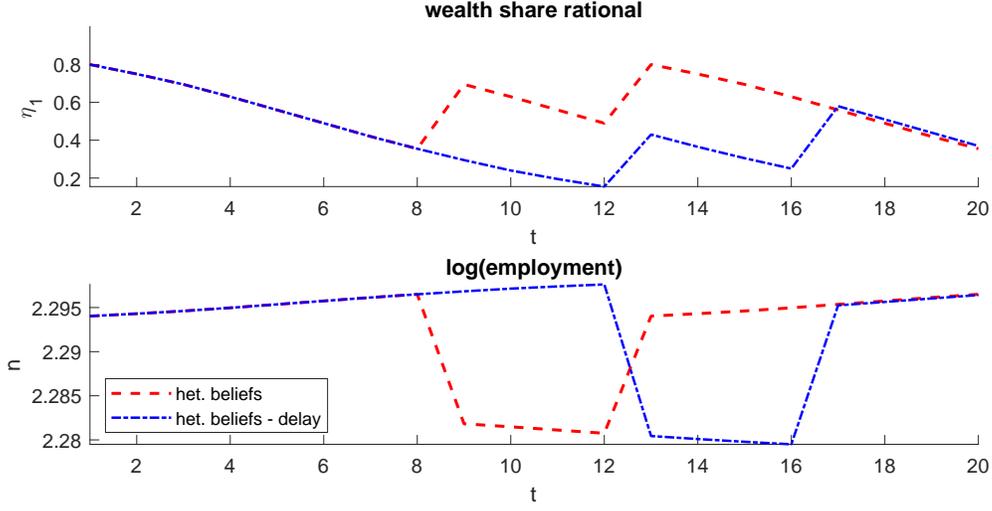


Figure 3: Asymmetric real business cycles

6.2 Flow of funds to investors

To illustrate how the liquidity flow to investors fixes the financial moments, we consider a rule-of-thumb supply of funds from the workers,

$$\hat{B}'(X, s) = \phi_s(\hat{w}h + R\hat{B}),$$

where ϕ_s is the state-dependent propensity to accumulate out of current income plus current wealth, $\hat{w}h + R\hat{B}$. In the Appendix B.2, we show how to solve the model numerically for this case. Recall that $\hat{w}_t = \xi h_t^{1+\nu}$, and note that both the wage bill $\xi h_t^{1+\nu}$ and labor h_t are pre-determined variables that can be easily recovered from pre-determined profits \hat{E} . By plugging the equation above into equation (10), one obtains

$$\hat{q}(X, s) = \frac{1}{1-\beta} \left[\beta \hat{E} + \phi_s \xi h_t^{1+\nu} + (\phi_s - \beta) R \hat{B} \right].$$

This expression shows that as the cycle is amplified during the good state, higher wages and higher flow of funds to the speculators push the price per share up. In addition, $\phi_H > \phi_L$ enhances the chances of getting a pro-cyclical P/E ratio. This assumption could be justi-

fied by a precautionary motive as the worker faces incomplete markets, or by consumption smoothness as in Guvenen [2009] who assumes that non-stockholders have a lower IES than stockholders. Note that the supply of funds does not depend on the risk-free interest rate $R_b(X, s)$, which tends to be higher in bad states as in standard RBC models. Hence, if savings were very sensitive to $R_b(X, s)$, there would be a counteracting force pushing the price per share down during the boom. As long as this substitution effect is not that strong, the rule-of-thumb supply of funds should be qualitatively in line with its counterpart derived from optimization behavior on the side of the worker.

Figures 4 and 5 are analogous to Figures 1 and 2, except that they incorporate the aforementioned supply of funds schedule and we report the evolution of the P/E ratio instead of the bubble component. We set $\phi_H = \beta = 0.99$ and $\phi_L = 0.988$. These figures imply that the worker holds 23.7 percent of the total wealth in the steady-state with $\eta_1 = 1$.

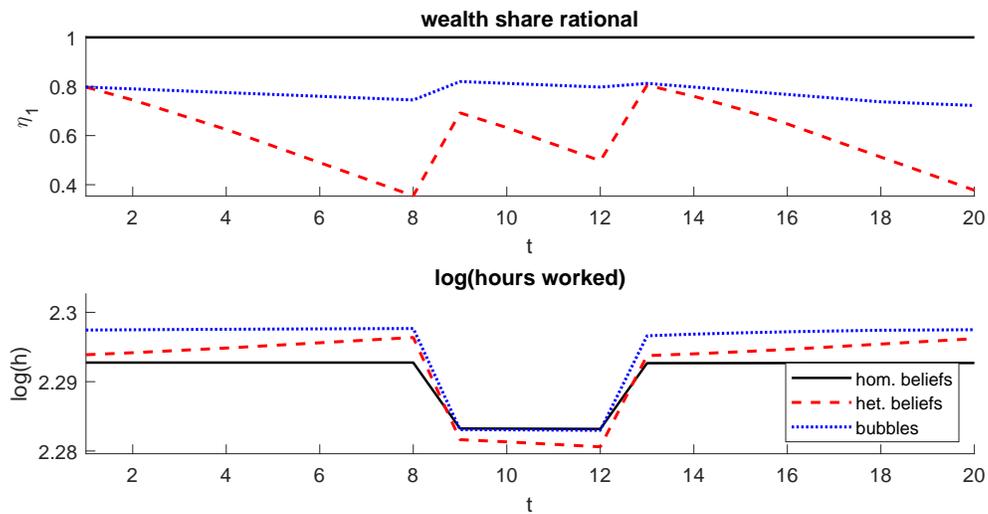


Figure 4: Real business cycle: supply of funds

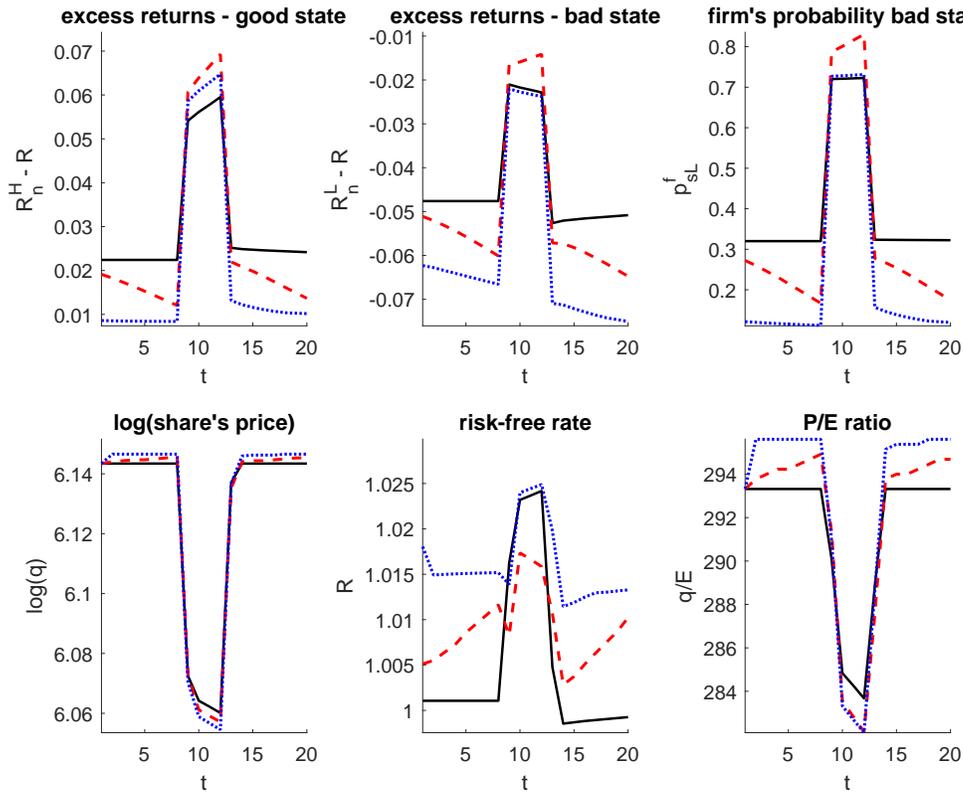


Figure 5: Financial market outcomes: supply of funds

The figures are purely illustrative. They show that a proper supply of funds can fix, at least qualitatively, some of the financial moments the model without supply of funds missed. Indeed, although barely visible, the price per share increases (declines) conditional on being in a good (bad) state. Also, not only the P/E ratio becomes pro-cyclical, but heterogeneous beliefs and speculation amplify its cycle. Nonetheless, the business cycle amplification is very similar in both cases with and without supply of funds.

7 Quantitative exploration

In this section we conduct a quantitative exploration. The idea is to understand the potential that speculation and bubbles have to magnify business cycles fluctuations. As in Hall [2017], the transmission mechanism from asset prices to business cycles relies on discount-

ing. Within the spirit of the first generation of real business cycle models, we abstract from financial frictions and pecuniary externalities, as in Dávila and Korinek [2017], or sticky prices and aggregate demand externalities as in Caballero and Simsek [2019]. If anything, this quantitative exploration should be seen as a lower bound on the potential effects of speculation. Also, for simplicity, we assume $\hat{B}'(X, s) = \hat{B} = 0$, and we are aware that we are missing qualitatively some financial moments, but not those relevant for the mechanics of the model.

The calibration is standard. We set the model period to quarters, so $\beta = 0.991$. Regarding the production function, we assume a labor share of $\alpha = 2/3$. The good state is associated with quarterly growth of 1.2 percent, $g_H = 1.012$, where the bad state with a recession of -0.4 percent, $g_L = 0.996$. The average duration of a recession (expansion) is four (ten) quarters, implying that $p_{LL} = 0.75$ ($p_{HH} = 0.90$). These figures are in congruence with rough calculations using the NBER recession dates, or with more elaborated estimations such as Hamilton [1989]. The Frisch elasticity is 2, meaning that $\nu = 0.5$. We set the relative risk aversion to $\gamma = 1$. ξ is just a scale parameter, normalized to one. The aforementioned parameters are fixed throughout all specifications.

Absent the time-to-build assumption, hours decline by 1.91 percent once the recession hits, and the standard deviation of (log) hours is 0.86 percent, in line with the outcomes of early real business cycle models, e.g. Cooley and Prescott [1995]. These figures are upper bounds on the effects once we impose the time-to-build assumption. Indeed, equation (2) shows that labor is employed taking into consideration expected TFP growth, which is higher (lower) than \bar{g}_L (\bar{g}_H) at the bad (good) state. As an example consider the steady-state with $\eta_1 = 1$. In this case, firm's adjusted probabilities are $p_{LL}^f = 0.759$ and $p_{HH}^f = 0.896$, implying that the standard deviation of (log) hours is 0.57 percent, and hours decline by 1.25 percent once the recession hits. Hence, the maximal amplification due to heterogeneous beliefs and speculation is obtained by pushing both p_{LL}^f and p_{HH}^f closer to one. At that bound, this increases the standard deviation by a factor of 1.51 ($0.86/0.57$) and the amplitude of the cycle by 1.53 ($1.91/1.25$).

To gauge the role of heterogeneous beliefs, speculation and bubbles we ran different

specifications with short-selling, $\kappa = \infty$, and without short-selling at all, $\kappa = 0$. Our simulations suggest that even when $\kappa = 0$, the economy converges to the steady-state in which $\eta_1 = 1$ and, thus, short-selling constraints cannot be binding. We also vary the diagnostic parameter θ from 0 to 2. Figure 6 shows how p_{LL}^2 and p_{HH}^2 vary with θ .

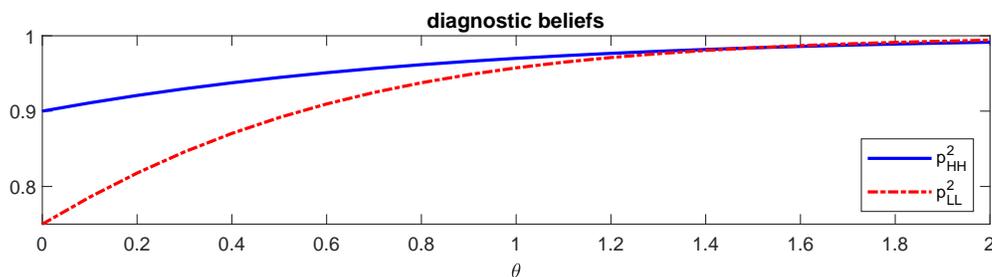


Figure 6: Diagnostic beliefs

Conditional on a recession, recession is very diagnostic, and thus subject to lots of disagreement. Bordalo et al. [forthcoming], Bordalo et al. [2018] and Bordalo et al. [2019] estimate θ to be around 0.9, 0.9 and 0.6 in the context of forecasting stock returns, explaining credit cycles, and forecasting macroeconomic variables, respectively. Hence, even moderate values of θ can distort substantially subjective probabilities of remaining in a recession. For instance, if $\theta = 0.4$, the diagnostic investor believes the economy will remain in a recession with a probability of 87 percent as opposed to the true probability of 75 percent. As θ further increases, such probability even surpasses the subjective probability of remaining in a boom, despite the boom state being more persistent. In a few words, diagnostic investors get very scared once the recession hits.

Figure 7 showcases the amplification in hours decline from the peak to the trough of the cycle, for a one-year recession after two-years and a half expansion, in line with the average cycle encoded in the Markov chain. In particular, the top panel shows how the decline in hours is amplified (with respect to an economy with $\eta_1 = 1$) after a fraction of investors become diagnostic. The shocks to population reduce η_1 to 0.2, 0.5 and 0.8 at the first period of expansion. As in the aforementioned narrative account for the cycles, one can interpret such large shocks as a big waves of optimism. The dashed-lines consider an

economy without short-selling constraints ($\kappa = \infty$) and, thus, without bubbles. If $\theta = 1$, for example, labor decline is amplified by 36, 24, and 11 percent for shocks that bring η_1 to 0.2, 0.5 and 0.8, respectively.¹² Given that the upper bound for such amplification is nearly 50 percent, we interpret those figures as sizable.

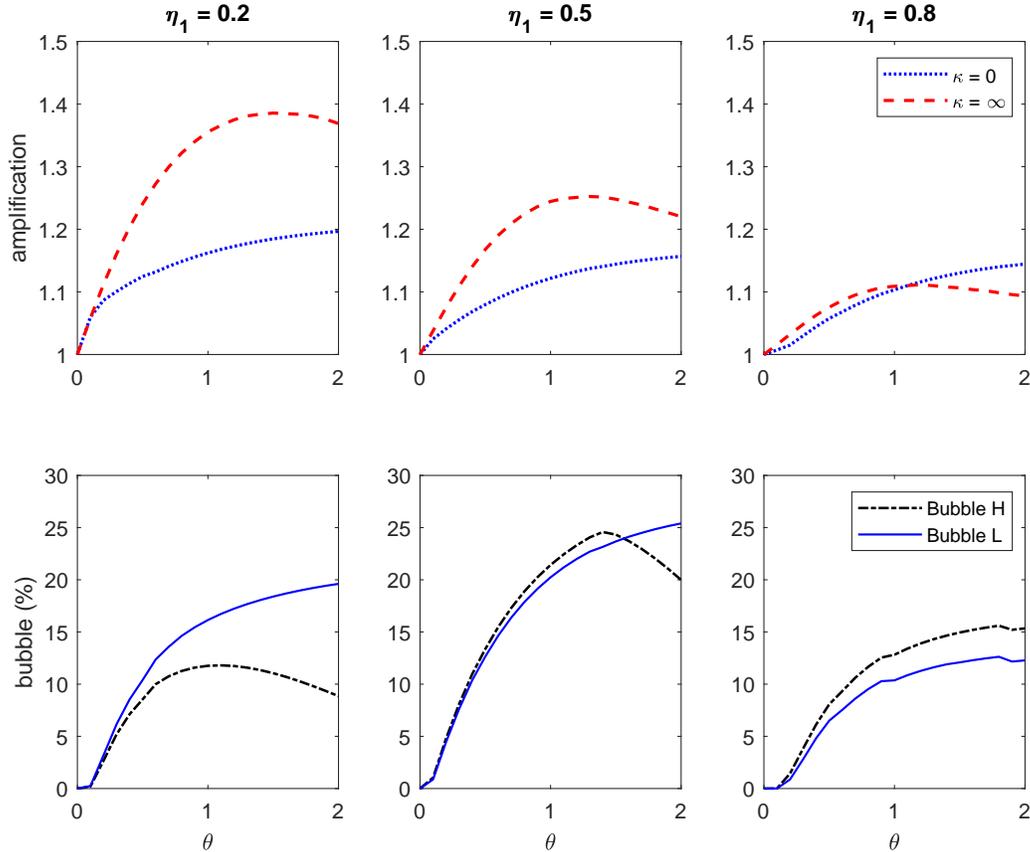


Figure 7: Amplification of labor responses

The effect of short-selling constraints ($\kappa = 0$, dotted-lines) is non monotone. Once tight short-selling constraints are imposed, the amplification may be higher or lower than the one in the economy with loose constraints. Intuitively, bubbles push excess returns down and employment up both at the good and at the bad states, as well as at both the steady-state when $\eta_1 = 1$ and along the transition triggered by the diagnostic shock. Hence the

¹²We consider $\theta = 1$ as the benchmark value for the diagnostic parameter. As our model does not feature a representative agent, we push a bit up the upper end of the aforementioned estimations for θ as some rational agents might countervail diagnostic expectations in the aggregate.

overall effect is ambiguous.

For intermediate values of θ , not only amplification is still sizable, but so are the bubble component (the log-difference between the stock price and the largest subjective valuation). The bottom panel of Figure 7 shows the bubble component before and after the bust. In particular, the dashed-lines represent the bubble component at the last period of expansion, whereas the filled-lines at the first period of recession. For $\theta = 1$, the bubble component ranges from 10 percent to 20 percent. Interestingly, depending on the size of the shock, it can be higher at low states. Hence, not always the model delivers a key feature of the narrative of boom-bust cycles that recessions are associated with the burst of bubbles. It depends on the size of the shock as well as on the degree of disagreement. As an example, if $\theta = 1$ and the shock brought η_1 to 0.8, the “burst” of the bubble was moderate as the bubble component in asset prices was reduced from 12.8 to only 10.4 percent after the recession hit the economy.

With respect to an economy with unlimited short-selling, on one hand, short-selling constraints may reduce the amplification of labor responses in the first cycle after a diagnostic shock. On the other hand, short-selling constraints may produce greater labor volatility, by prolonging the periods of disagreement in the economy as wealth becomes a slower moving variable. To quantify this claim, we simulate for $\theta = 1$ ten thousands paths for economies that were in steady-state and were hit by diagnostic shocks that brought η_1 to 0.2, 0.5 and 0.8. The top panel of Figure 8 plots the mean across paths of twenty-years rolling-window of the standard deviation of (log) hours for economies with (dotted-lines) and without (dashed-lines) short-selling constraints, relative to the steady-state with $\eta_1 = 1$. The bottom panel plots the respective evolution of the wealth share (also the mean across paths).

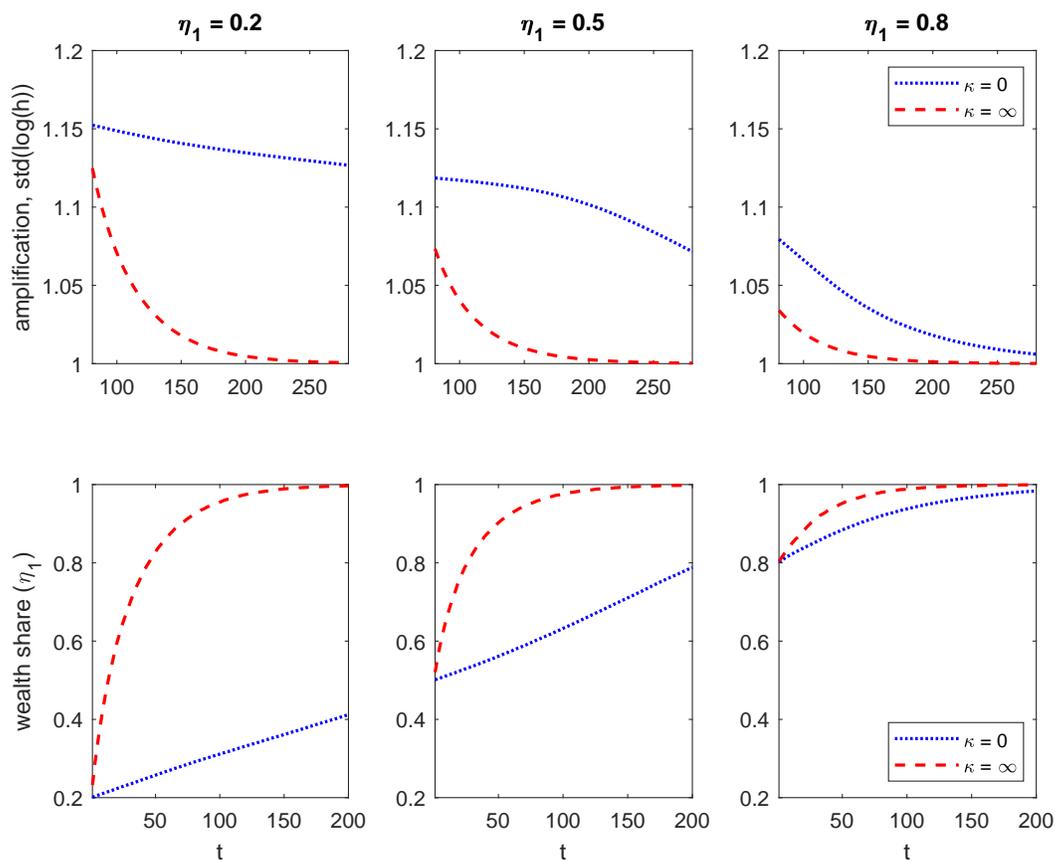


Figure 8: Amplification of the volatility of hours

The volatility of hours increases in both economies, with and without short-selling constraints, relative to the steady-state with $\eta = 1$. Nonetheless, the amplification is larger in the former. Intuitively, as wealth tends to converge to the steady-state with $\eta_1 = 1$ at a slower pace in the economy with short-selling constraints, this economy sustains a lengthier amplification of labor responses. This translates into more volatile cycles. The effects are sizable taking into consideration that there is an upper bound of a 50 percent increase in hours volatility. Indeed, the standard deviation of labor hours can be as large as 15, 12, and 8 percent when the diagnostic shock η_1 is brought to 0.2, 0.5 and 0.8, respectively.

8 Conclusion

In this paper we adapt a standard real business cycle model to incorporate the speculation-driven business cycles view. In particular, we spell out the key model ingredients that are sufficient to comprehensively account for such view. In what follows we summarize some of our findings, and point out some directions for future research.

We derive an equilibrium equation that relates the price per share to current profits and the flow of funds to investors. Given that markets are segmented, the price per share should reflect fluctuations in the net supply of funds to the financial sector. This expression is valid for any configuration of beliefs, any value for the risk aversion, and independent of the presence of short-selling (or borrowing) constraints. The price that absorbs speculation, whether the price per share or the risk-free interest rate, depends on the amount of funds supplied at good and bad states. In particular, a bubble component could arise even without a surge in the stock price due to higher discounting.

This result speaks to the leaning against the wind debate (see Gourio et al. [2018] for a recent contribution). Suppose the government can control the risk-free rate. In our setup this could be accomplished through fiscal or macro-prudential policy. If heterogeneous beliefs and short-selling constraints are the driving forces behind bubbles in financial markets, this result suggests that the government cannot fully “burst” the bubble, but it can decide to what extent the stock price or the risk-free rate should absorb speculation. Hence, the answer hinges on the trade-offs of stabilizing one price at the cost of making the other one fluctuate more. Hence, it depends crucially on the frictions assumed: sticky prices, fire-sales externality, working capital constraint, etc. We leave these possible explorations for future research.

With respect to the amplification of the business cycle, a few conclusions are worth emphasizing. First, as optimistic (pessimistic) investors tend to accumulate wealth during the expansions (recessions), real business cycles are amplified. Second, the longer is the boom period, the more severe is the bust, an immediate corollary of the first result. Although the modeling of the worker is stylized, these results hold for any supply of funds schedule, including the one derived from optimal behavior. Finally, if short-selling constraints are

imposed, such that the economy becomes prone to bubbles, simulations suggest we get further amplification of the cycle at good states, but attenuation in bad states.

A quantitative exercise within a simplified framework suggests that not only the effects of heterogeneous beliefs and speculation, but also bubbles can be sizable. The volatility of hours can be amplified up to 15 percent, and bubbles imply that the stock price could be up to 20 percent higher. The transmission mechanism relies on the overall willingness to bear risk in the economy, and except for the time-to-build assumption regarding labor, no other frictions are necessary. By construction, cycles cannot be amplified more than 50 percent in our quantitative exploration. Hence we see these quantitative results as a sizable primary and direct effect due to speculation, that should be arguably amplified were other frictions, such as sticky prices or fire-sales externalities, accounted for.

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Appendices

A Working out wealth evolution (4) and FOC (5) for S=2

In this appendix, given $q(X, s)$, $R_b(X, s)$, $R_n(X, s)$, $\psi(X, s, s')$ and $v_i(X, s, s') = V_i(\psi(X, s, s'), s')$, we derive quasi-closed form solutions for the optimal portfolio choices when $S = 2$. Equation (5) can be rewritten as

$$\mathbb{E}_{i,s} \left[\frac{(v_i(X, s, s'))^{1-\gamma} [R_b(X, s) - R_n(X, s, s')]}{\left[R_n(X, s, s')(1 - \tilde{b}_i(X, s)) + R_b(X, s)\tilde{b}_i(X, s) \right]^\gamma} \right] = 0.$$

Under the assumption of two shocks, say $s \in \{L, H\}$,

$$p_{s,L}^i \left[\frac{(v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)]}{\left[R_n(X, s, L)(1 - \tilde{b}_i(X, s)) + R_b(X, s)\tilde{b}_i(X, s) \right]^\gamma} \right] = -p_{s,H}^i \left[\frac{(v_i(X, s, H))^{1-\gamma} [R_b(X, s) - R_n(X, s, H)]}{\left[R_n(X, s, H)(1 - \tilde{b}_i(X, s)) + R_b(X, s)\tilde{b}_i(X, s) \right]^\gamma} \right],$$

which can be rewritten as

$$\frac{\{p_{s,L}^i (v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)]\}^{\frac{1}{\gamma}}}{R_n(X, s, L) + [R_b(X, s) - R_n(X, s, L)]\tilde{b}_i(X, s)} = \frac{\{p_{s,H}^i (v_i(X, s, H))^{1-\gamma} [R_n(X, s, H) - R_b(X, s)]\}^{\frac{1}{\gamma}}}{R_n(X, s, H) - [R_n(X, s, H) - R_b(X, s)]\tilde{b}_i(X, s)},$$

by further developing the expression above,

$$\begin{aligned} & \left\{ p_{s,L}^i (v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)] \right\}^{\frac{1}{\gamma}} \{ R_n(X, s, H) - [R_n(X, s, H) - R_b(X, s)]\tilde{b}_i(X, s) \} = \\ & = \left\{ p_{s,H}^i (v_i(X, s, H))^{1-\gamma} [R_n(X, s, H) - R_b(X, s)] \right\}^{\frac{1}{\gamma}} \{ R_n(X, s, L) + [R_b(X, s) - R_n(X, s, L)]\tilde{b}_i(X, s) \}. \end{aligned}$$

Hence, after collecting terms,

$$\tilde{b}_i(X, s) = \frac{\{p_{s,L}^i (v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)]\}^{\frac{1}{\gamma}} R_n(X, s, H) - \{p_{s,H}^i (v_i(X, s, H))^{1-\gamma} [R_n(X, s, H) - R_b(X, s)]\}^{\frac{1}{\gamma}} R_n(X, s, L)}{\{p_{s,H}^i (v_i(X, s, H))^{1-\gamma} [R_n(X, s, H) - R_b(X, s)]\}^{\frac{1}{\gamma}} [R_b(X, s) - R_n(X, s, L)] + \{p_{s,L}^i (v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)]\}^{\frac{1}{\gamma}} [R_n(X, s, H) - R_b(X, s)]} \quad (12)$$

Now, by plugging the equation above in equation (4), i.e. $\frac{\bar{g}_s \alpha'(X, s, s')}{\beta a} = R_n(X, s, s') +$

$[R_b(X, s) - R_n(X, s, s')] \tilde{b}_i(X, s)$, and working it out, one obtains that

$$\begin{aligned}
& R_n(X, s, s') + [R_b(X, s) - R_n(X, s, s')] \times \\
& \times \frac{\left\{ p_{s,L}^i(v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)] \right\}^{\frac{1}{\gamma}} R_n(X, s, H) - \left\{ p_{s,H}^i(v_i(X, s, H))^{1-\gamma} [R_n(X, s, H) - R_b(X, s)] \right\}^{\frac{1}{\gamma}} R_n(X, s, L}}{\left\{ p_{s,H}^i(v_i(X, s, H))^{1-\gamma} [R_n(X, s, H) - R_b(X, s)] \right\}^{\frac{1}{\gamma}} [R_b(X, s) - R_n(X, s, L)] + \left\{ p_{s,L}^i(v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)] \right\}^{\frac{1}{\gamma}} [R_n(X, s, H) - R_b(X, s)]}} = \\
& = \frac{\left\{ p_{s,s'}^i(v_i(X, s, s'))^{1-\gamma} [R_b(X, s) - R_n(X, s, s')] \right\}^{\frac{1}{\gamma}} [R_b(X, s) - R_n(X, s, L)] R_n(X, s, H) + \left\{ p_{s,s'}^i(v_i(X, s, s'))^{1-\gamma} [R_b(X, s) - R_n(X, s, s')] \right\}^{\frac{1}{\gamma}} [R_n(X, s, H) - R_b(X, s)] R_n(X, s, L}}{\left\{ p_{s,H}^i(v_i(X, s, H))^{1-\gamma} [R_n(X, s, H) - R_b(X, s)] \right\}^{\frac{1}{\gamma}} [R_b(X, s) - R_n(X, s, L)] + \left\{ p_{s,L}^i(v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)] \right\}^{\frac{1}{\gamma}} [R_n(X, s, H) - R_b(X, s)]}} = \\
& = \frac{\left\{ p_{s,s'}^i(v_i(X, s, s'))^{1-\gamma} [R_b(X, s) - R_n(X, s, s')] \right\}^{\frac{1}{\gamma}} \{ [R_b(X, s) - R_n(X, s, L)] R_n(X, s, H) + [R_n(X, s, H) - R_b(X, s)] R_n(X, s, L) \}}{\left\{ p_{s,H}^i(v_i(X, s, H))^{1-\gamma} [R_n(X, s, H) - R_b(X, s)] \right\}^{\frac{1}{\gamma}} [R_b(X, s) - R_n(X, s, L)] + \left\{ p_{s,L}^i(v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)] \right\}^{\frac{1}{\gamma}} [R_n(X, s, H) - R_b(X, s)]}} = \\
& = \frac{\left\{ p_{s,s'}^i(v_i(X, s, s'))^{1-\gamma} [R_b(X, s) - R_n(X, s, s')] \right\}^{\frac{1}{\gamma}} R_b(X, s) [R_n(X, s, H) - R_n(X, s, L)]}{\left\{ p_{s,H}^i(v_i(X, s, H))^{1-\gamma} [R_n(X, s, H) - R_b(X, s)] \right\}^{\frac{1}{\gamma}} [R_b(X, s) - R_n(X, s, L)] + \left\{ p_{s,L}^i(v_i(X, s, L))^{1-\gamma} [R_b(X, s) - R_n(X, s, L)] \right\}^{\frac{1}{\gamma}} [R_n(X, s, H) - R_b(X, s)]}}. \tag{13}
\end{aligned}$$

To obtain the expressions reported in Lemma 2, just plug $\gamma = 1$ into equations (12) and (13) above.

B Equilibrium equations and numerical algorithm

Throughout the main text, we consider $X = \{\hat{E}, \{\eta_i\}_{i=1}^{I-1}, R\hat{B}\}$ because it facilitates exposition and intuition. However, to implement the model numerically, we propose a change of the state space that renders the algorithm more efficient.

First, instead of using $R\hat{B}$ as a state variable, we consider $\eta_w = \frac{R\hat{B}}{\sum_i \mu_i \hat{a}_i + R\hat{B}}$. Recall that $\sum_i \mu_i \hat{a}_i + R\hat{B} = \hat{q}(X, s) + \hat{E}$. Hence by substituting $R\hat{B}$ for $\eta_w(\hat{q}(X, s) + \hat{E})$ in the model, we can characterize the equilibrium in terms of η_w , which is bounded in between zero and one, rather than $R\hat{B}$. Note that we do not change the definition of $\eta_i = \frac{\mu_i \hat{a}_i}{\sum_i \mu_i \hat{a}_i}$, which represents wealth shares among investors only.

Second, instead of using the current de-trended profits \hat{E} , we consider the **previous** expected value for TFP, computed using firm's adjusted beliefs. Call it ε . Hence, with slight abuse of notation, current de-trended profits are given by

$$\hat{\pi}(X, s) = \frac{1}{\xi^{\frac{\alpha}{1+\nu-\alpha}}} \left\{ \bar{g}_s (\alpha \varepsilon)^{\frac{\alpha}{1+\nu-\alpha}} - (\alpha \varepsilon)^{\frac{1+\nu}{1+\nu-\alpha}} \right\}.$$

Note that by knowing s and ε , one can compute $\hat{E} = \hat{\pi}(X, s)$. The advantage of using ε instead of \hat{E} is twofold. First, it is bounded by the lowest and the highest TFP growth shocks. Second, it renders a more efficient numerical algorithm as the updating rule for ε , $\varepsilon'(X, s) = \mathbb{E}_{X,s}^f[g]$, does not depend on the future exogenous state s' .

Again, with a slight abuse of notation, let $X = \{\varepsilon, \{\eta_i\}_{i=1}^{I-1}, \eta_w\}$. Given this change of variables, a simple algebra reveals that the key equation (10) in the main text now reads

$$\hat{q}(X, s) = \frac{\beta(1 - \eta_w)\hat{\pi}(X, s) + \bar{g}_s \hat{B}'(X, s)}{1 - \beta(1 - \eta_w)}. \quad (14)$$

Under the assumption that $S = 2$, equation (12) in the Appendix A solves for the optimal portfolio weights $\tilde{b}_i(X, s)$ as functions of $\hat{q}(X, s)$, $R_b(X, s)$, $R_n(X, s)$, $\psi(X, s, s')$ and $v_i(X, s, s') = V_i(\psi(X, s, s'), s')$. This is the only part of the solution method that the assump-

tion of $S = 2$ kicks in. The market clearing condition for bonds reads:

$$\bar{g}_s \hat{B}'(X, s) = - \frac{\beta(\hat{\pi}(X, s) + \bar{g}_s \hat{B}'(X, s) - \eta_w(\hat{q}(X, s) + \hat{\pi}(X, s)))}{1 - \beta} \sum_i \tilde{b}_i(X, s) \eta_i.$$

If $\gamma \neq 1$, then the value function $V_i(X, s)$ satisfies the following recursion:

$$\begin{aligned} \ln V_i(X, s) &= (1 - \beta) \ln(1 - \beta) + \beta \ln \beta + \\ &+ \frac{\beta}{1 - \gamma} \ln \left(\left\{ \mathbb{E}_{i,s} \left[\left(V_i(\psi(X, s, s'), s') \left[R_n(X, s, s')(1 - \tilde{b}_i(X, s)) + R_b(X, s) \tilde{b}_i(X, s) \right] \right)^{1-\gamma} \right] \right\} \right). \end{aligned}$$

If $\gamma = 1$ the solution simplifies as portfolios weights, $\tilde{b}_i(X, s)$, cease to depend on $v_i(X, s, s')$, and thus, the recursion above when $\gamma \rightarrow 1$ is immaterial for the solution.

The law of motion $X' = \psi(X, s, s')$ is implicitly defined by

$$\varepsilon'(X, s) = \mathbb{E}_{X,s}^f[g];$$

$$\eta'_i(X, s, s') = \frac{\left[R_n(X, s, s')(1 - \tilde{b}_i(X, s)) + R_b(X, s) \tilde{b}_i(X, s) \right] \eta_i}{\sum_{i=1}^I \left[R_n(X, s, s')(1 - \tilde{b}_i(X, s)) + R_b(X, s) \tilde{b}_i(X, s) \right] \eta_i};$$

$$\eta'_w(X, s, s') = \frac{R_b(X, s) \hat{B}'(X, s)}{q(\psi(X, s, s'), s') + \hat{\pi}(\psi(X, s, s'), s')}.$$

Recall that stock returns are given by

$$R_n(X, s, s') = \frac{\hat{q}(\psi(X, s, s'), s') + \hat{\pi}(\psi(X, s, s'), s')}{\hat{q}(X, s)},$$

whereas the stochastic discount factors are given by

$$SDF_i(X, s, s') = \frac{(v_i(X, s, s'))^{1-\gamma}}{\left[R_n(X, s, s')(1 - \tilde{b}_i(X, s)) + R_b(X, s) \tilde{b}_i(X, s) \right]^\gamma}.$$

Finally, firm's adjusted beliefs are given by

$$p_{ss'}^f(X) = \sum_i \omega_i(X, s) \left[\frac{SDF_i(X, s, s') p_{ss'}^i}{\sum_{s'} SDF_i(X, s, s') p_{ss'}^i} \right],$$

where we assume that weights are given by

$$\omega_i(X, s) = \frac{\mathbf{I}_{\{\tilde{b}_i(X, s) < 1\}} \eta_i (1 - \tilde{b}_i(X, s))}{\sum_i \mathbf{I}_{\{\tilde{b}_i(X, s) < 1\}} \eta_i (1 - \tilde{b}_i(X, s))}.$$

After assuming a functional form for the supply of funds to the investors, $B'(X, s)$, one can solve numerically the model in the computer.

We propose the following numerical algorithm to compute globally the equilibrium. Discretize ε , $\{\eta_i\}_{i=1}^{I-1}$ and η_w , fix $\hat{B}'(X, s)$, and conjecture the law of motion $X' = \psi(X, s, s')$. The idea is to iterate over $\psi(X, s, s')$ until convergence is reached. With $\hat{B}'(X, s)$ and a guess for $\psi(X, s, s')$ at hand, one can compute $\hat{q}(X, s)$ and $R_n(X, s, s')$. Hence, within each iteration, use the bisection method to find $R_b(X, s)$ that clears the bonds market, keeping in mind that if $\gamma \neq 1$, one also needs to iterate over $V_i(X, s)$ inside this inner loop to compute $\tilde{b}_i(X, s)$. With $R_n(X, s, s')$, $R_b(X, s)$ and $\tilde{b}_i(X, s)$ at hand, one can update $\eta'_i(X, s, s')$, $\eta'_w(X, s, s')$ and $\varepsilon'(X, s)$ to obtain a new guess $\psi(X, s, s')$ for the next iteration.

This numerical algorithm is easily malleable if we impose short-selling or debt constraints: $\tilde{b}_i(X, s) \in [-\lambda, 1 + \kappa]$, with $\lambda \geq 0$ and $\kappa \geq 0$. All we need is to replace $\tilde{b}_i(X, s)$ obtained in equation (12) after working out the FOCs by $-\lambda$ or $1 + \kappa$ accordingly if the solution is outside those bounds.

B.1 Specialization: $\gamma = 1$, $\hat{B}'(X, s) = \hat{B} = 0$ and $I = 2$ (as well as $S = 2$)

As we argued above, if $\gamma = 1$, the solution simplifies as $\tilde{b}_i(X, s)$ and $SDF_i(X, s, s')$ cease to depend on $v_i(X, s, s')$ (see expressions obtained in Lemma 2 in the main text), avoiding an inner loop to compute $v_i(X, s, s')$ recursively. As discussed in the main text, under $\gamma = 1$ and $S = 2$ with $s \in \{L, H\}$, we obtain the following close form for the evolution of wealth

shares,

$$\eta'_i(X, s, s') = \frac{p_{ss'}^i \eta_i}{\sum_{i=1}^I p_{ss'}^i \eta_i}.$$

Also, under these assumptions, firm's adjusted beliefs become a simple function of stock returns and the risk-free rate,

$$p_{sL}^f(X, s) = \frac{R_n(X, s, H) - R_b(X, s)}{R_n(X, s, H) - R_n(X, s, L)} \quad \text{and} \quad p_{sH}^f(X, s) = \frac{R_b(X, s) - R_n(X, s, L)}{R_n(X, s, H) - R_n(X, s, L)}.$$

These formulas are valid for any supply of funds schedule, $\hat{B}'(X, s)$, and simplify the computation of the equilibrium as there is no need to iterate over $\eta'_i(X, s, s')$ and $v_i(X, s, s')$.

Now if we assume there is no supply of funds to the investors, $\hat{B}'(X, s) = \hat{B} = 0$, which implies $\eta'_w(X, s, s') = \eta_w = 0$, the solution for the price per share is simplified,

$$\hat{q}(X, s) = \frac{\beta \hat{\pi}(X, s)}{1 - \beta},$$

which implies the following expression for asset returns,

$$R_n(X, s, s') = \frac{\pi(\psi(X, s, s'), s')}{\beta \hat{\pi}(X, s)} = \frac{1}{\beta} \frac{\bar{g}_{s'}(\alpha \varepsilon'(X, s))^{\frac{\alpha}{1+\nu-\alpha}} - (\alpha \varepsilon'(X, s))^{\frac{1+\nu}{1+\nu-\alpha}}}{\bar{g}_s(\alpha \varepsilon)^{\frac{\alpha}{1+\nu-\alpha}} - (\alpha \varepsilon)^{\frac{1+\nu}{1+\nu-\alpha}}}.$$

In addition, the market clearing condition for bonds reads

$$\sum_{i=1}^I \eta_i \tilde{b}_i(X, s) = 0.$$

These expressions are valid for any value of γ , and $S = 2$ is only needed to compute $\tilde{b}_i(X, s)$. They facilitate the implementation of the equilibrium as, of course, we do not even need to track η_w as a state variable, reducing the dimensionality of the state space.

Finally, by assuming altogether $\gamma = 1$, $\hat{B}'(X, s) = \hat{B} = 0$ and $I = 2$ (as well as $S = 2$), one can use the expressions in Lemma 2 to work out the above market clearing condition for bonds, and reach an expression for $R_b(X, s)$ also as a function of stock returns.

$$R_b(X, s) = \frac{R_n(X, s, H)R_n(X, s, L)}{(\eta_1 p_{sL}^1 + \eta_2 p_{sL}^2)R_n(X, s, H) + (\eta_1 p_{sH}^1 + \eta_2 p_{sH}^2)R_n(X, s, L)}.$$

These extra assumptions make the numerical computation of the equilibrium significantly more efficient, as one only needs to iterate over $\varepsilon'(X, s)$ within a state space with lower dimensionality. In addition, there is no need to use the bisection method to find the $R_b(X, s)$ that clears the market.

Nonetheless, if we impose bounds on the portfolio decisions, $\tilde{b}_i(X, s)$, we lose the tractability gained from the assumption that $\gamma = 1$. Hence, to compute numerically the equilibrium, one also needs to iterate over $\eta_i(X', s, s)$ and run the bisection method to find $R_b(X, s)$ that clears the bonds market. But the state space is now reduced due to the assumption that $\hat{B}'(X, s) = \hat{B} = 0$.

B.2 Specialization: $\hat{B}'(X, s) = \phi_s(\hat{w}h + R\hat{B})$

We consider in this subsection the case $\hat{B}'(X, s) = \phi_s(\hat{w}h + R\hat{B})$. Recall that the algorithm outlined above computes the endogenous objects by taking $B'(X, s)$ as given. Also recall that the state space is represented by $X = \{\varepsilon, \{\eta_i\}_{i=1}^{I-1}, \eta_w\}$. Hence,

$$\hat{B}'(X, s) = \phi_s(\hat{w}(\varepsilon)h(\varepsilon) + \eta_w(\hat{q}(X, s) + \hat{\pi}(X, s))), \quad (15)$$

which coupled with equation (14) determines both $B'(X, s)$ and $q(X, s)$ for each (X, s) . In this linear case, it is straight forward to obtain the solution. Indeed, one can further develop (14) by plugging (15) into, and after rearranging terms, obtain the following expression:

$$\hat{q}(X, s) = \frac{(\beta(1 - \eta_w) + \bar{g}_s \phi_s \eta_w) \hat{\pi}(X, s) + \bar{g}_s \phi_s \hat{w}(\varepsilon) h(\varepsilon)}{1 - \beta(1 - \eta_w) - \bar{g}_s \phi_s \eta_w}.$$

By plugging this expression back into (15), one obtains $B'(X, s)$ as a direct function of the state space, and can solve the model by applying the algorithm outlined above.

If the supply of funds were not linear, an intermediate step in the algorithm would be necessary to solve for both $\hat{q}(X, s)$ and $\hat{B}'(X, s)$, simultaneously.

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