

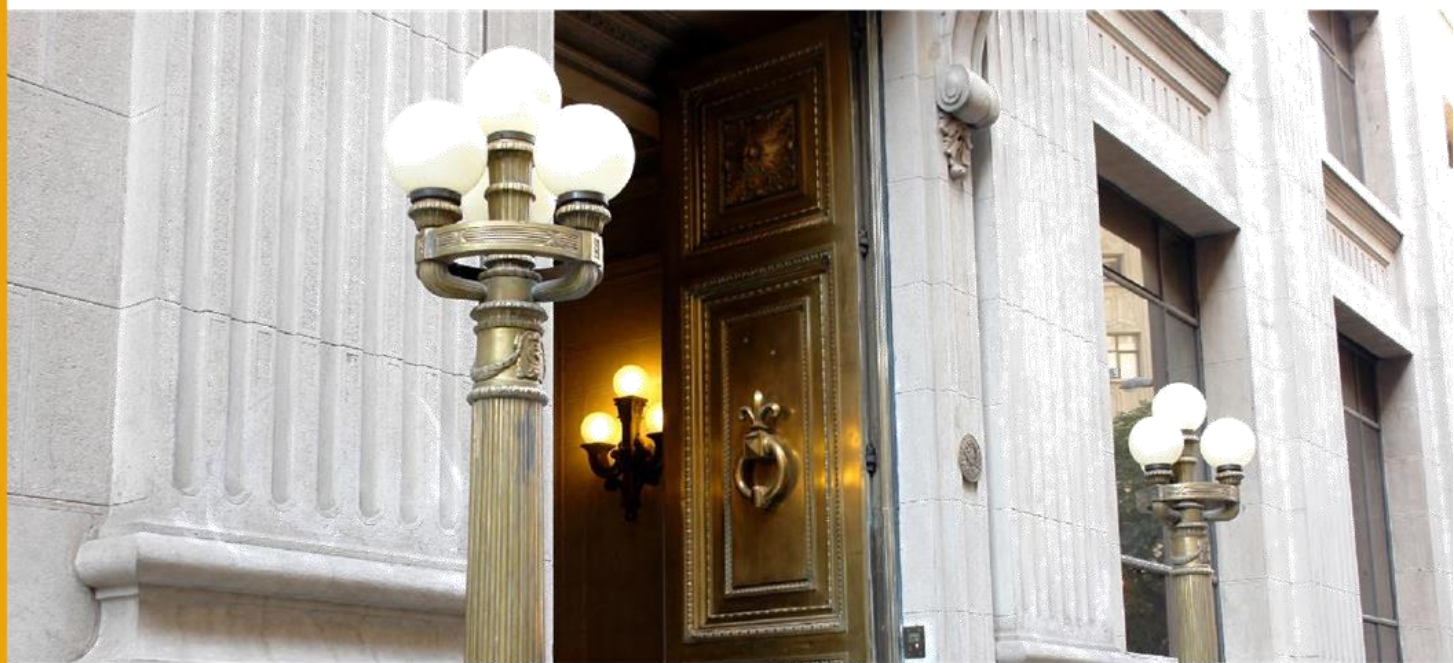
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A Technical Note on the Heston-Nandi Model

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A Technical Note on the Heston-Nandi Model*

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Resumen

En este artículo, derivamos en primer lugar una aproximación para la función generadora de momentos de los rendimientos acumulados generados por el modelo GARCH propuesto por Heston y Nandi (2000). Los cumulantes analíticos obtenidos son consistentes con los reportados en Mazzoni (2010), y los resultados numéricos confirman la calidad de la aproximación. En segundo lugar, proponemos una calibración que vincula los parámetros del modelo de Heston–Nandi con los del modelo GARCH asimétrico (AGARCH). Aunque reconocemos que se trata de modelos distintos, encontramos que la varianza y la asimetría presentan comportamientos similares.

Abstract

In this paper we first derive an approximation of the moment generating function of the cumulative returns generated by the GARCH model proposed by Heston and Nandi (2000). Analytical cumulants align those presented in Mazzoni (2010), and numerical results support the quality of the approximation. Second, we provide a calibration that relates the parameters of the Heston–Nandi model with those of the Asymmetric GARCH (AGARCH) model. Although we acknowledge that both models are different, we find that variance and skewness behave similarly.

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1 Introduction

Heston and Nandi (2000) propose a “GARCH version” of the continuous-time model introduced by Heston (1993). The latter is a popular stochastic volatility model among quantitative finance practitioners, and it can be easily calibrated with information from option prices (Gatheral, 2006; Rouah, 2013). A key feature of the Heston–Nandi (HN) model is that the Moment Generating Function (MGF) can be obtained in closed-form, after solving a recursion. Mazzoni (2010) proposes an approximation that avoids that recursion, providing a closed-form expression for the logarithm of the MGF (log-MGF) of the total returns (X_T). He then obtains analytical cumulants for X_T by computing the derivatives of the log-MGF.

In this note, we provide a closed-form expression for the log-MGF of the HN model based on a second-order expansion in one key parameter of the model (α), obtaining a polynomial log-MGF. Moments obtained from our approximation are explicit in the expression and also consistent with the ones reported in Mazzoni (2010). We conduct targeted Monte Carlo simulations to validate the quality of this approximation, concluding that it is accurate enough to replicate Mazzoni’s results. Finally, we explore the relationship between the HN model and the Asymmetric GARCH (AGARCH) model introduced by Engle (1990) and Engle and Ng (1993). We propose a parameter calibration of the HN model that matches the first three moments of the AGARCH one, but is not able to replicate the kurtosis.

The article is organized as follows. Section 2 provides a brief description of the Heston–Nandi and AGARCH models, and the main theoretical results. Section 3 presents Monte Carlo simulations, and Section 4 concludes.

2 Technical Section

In this section we describe the Heston–Nandi (HN) model and its properties, and provide an approximation for the logarithm of Moment Generating Function for cumulative returns. We conclude this section discussing the Asymmetric GARCH (AGARCH) model, which exhibits several features in common with the HN model.

2.1 Basics of HN model

The HN model states that the log-return (r_t) of the asset is equal to a constant (δ) plus a premium —proportional to the conditional variance (h_t)— and an error term that is heteroskedastic, as it is described by equation (1), where u_t is a standard Gaussian error term. The conditional variance is also time-variant, and its dynamic is determined by equation (2).¹

$$r_t = \delta + \lambda h_t + \sqrt{h_t} u_t, \quad (1)$$

$$h_{t+1} = \omega + \alpha \left(u_t - \gamma \sqrt{h_t} \right)^2 + \beta h_t. \quad (2)$$

Note that the model defined by equations (1) and (2) belongs to the GARCH family because it has a time-variant variance which is updated with past error terms (Engle, 1982). Moreover, for non-zero λ , the model features an in-mean component, which was introduced into GARCH model by Engle (1990) and Engle and Ng (1993) to capture the risk-premium, empirically observed in financial assets.²

¹Parameters are constrained as follows $\beta > 0$, $\omega + \alpha > 0$, and $\alpha\gamma^2 + \beta < 1$.

²The model can represent both the physical and the risk-neutral measures, with the latter being used to price options.

Cumulative total returns, between $t = 0$ and $t = T$, can be obtained by iterating (1):

$$X_T \equiv \sum_{t=0}^{T-1} r_{t+1} = \delta T + \lambda \sum_{t=0}^{T-1} h_{t+1} + \sum_{t=0}^{T-1} \sqrt{h_{t+1}} u_{t+1}. \quad (3)$$

The distribution of X_T is unknown for $T > 1$; however, Heston and Nandi (2000) show that the moment generating function of the terminal price (S_T) could be expressed as $M_t(\varphi) \equiv E_t(S_T^\varphi) = S_t^\varphi \exp(A_t + h_{t+1}B_t)$, where:

$$A_t = A_{t+1} + \varphi\delta + \omega B_{t+1} - \frac{1}{2} \log(1 - 2\alpha B_{t+1}), \quad (4)$$

$$B_t = \varphi(\gamma + \lambda) - \frac{\gamma^2}{2} + \beta B_{t+1} + \frac{0.5(\varphi - \gamma)^2}{1 - 2\alpha B_{t+1}}, \quad (5)$$

In the above expression, S_t represents the current price, and $A_T = B_T = 0$ are the terminal conditions.

2.2 Time-variant variances

We note that (2) can alternatively be expressed as $h_{t+1} = (\omega + \alpha) + (\alpha\gamma^2 + \beta)h_t + \alpha e_t$, where $e_t = (u_t^2 - 1) - 2\gamma u_t \sqrt{h_t}$ is a zero-mean error term. Thus, we can write (2) as: $h_{t+1} = h_\infty + (h_t - h_\infty)\rho + \alpha e_t$, where $\rho = \alpha\gamma^2 + \beta$ is the persistence of the conditional variance and $h_\infty = (\omega + \alpha)/(1 - \rho)$ is the long-run variance, and for future references we consider its square root which is the long-run-volatility (LRV).³

Expressing the dynamic of conditional variance in this way resembles a stochastic volatility (SV) model. To analyze this, we assume, for simplicity, that $u_t^2 \approx 1$, and $\eta = -2\alpha\gamma$ which implies the following expression $h_{t+1} = h_\infty + (h_t - h_\infty)\rho + \eta\sqrt{h_t}u_t$, providing a

³The covariance between h_{t+1} and returns is $\text{Cov}_{t-1}(h_{t+1}, r_t) = -2\alpha\gamma h_t$, thus the correlation between returns and h_{t+1} 's depends on the sign of γ . Also, the volatility of h_{t+1} is $\text{Vol}_{t-1}(h_{t+1}) = \sqrt{2 + 4\gamma^2 h_t \alpha}$.

convenience expression for an SV model in discrete-time.⁴

We note that when $\eta = 0$, conditional variances are time-variant yet deterministic. Building on this result and taking h_1 as a given initial variance, the conditional variance can then be solved recursively as: $h_{t+1} = h_\infty + (h_1 - h_\infty)\rho^t$; therefore, the average of (deterministic) variances, between $t = 0$ and $t = T$, is:

$$\bar{h} = \frac{1}{T} \sum_{t=0}^{T-1} h_{t+1} = h_\infty + \frac{(h_1 - h_\infty)}{T} \left(\frac{1 - \rho^T}{1 - \rho} \right). \quad (6)$$

Under this case ($\eta = 0$), total returns are normally distributed with variance $\bar{h}T$ due to the fact that time-variant variances are deterministic and uncorrelated with returns.

The previous result is no longer valid under general cases (i.e. $\eta > 0$); however, for small η , X_T can be approximated by a normal distribution with mean $(\delta + \lambda\bar{h})T$ and variance $(\bar{h} + \vartheta\tilde{h})T + O(\eta^2)$, where $\vartheta = 2\lambda\eta/(1 - \rho)$;⁵ see Appendix B:

$$\begin{aligned} \tilde{h} &= \frac{1}{T} \sum_{t=0}^{T-1} [1 - \rho^{(T-1)-t}] [h_\infty + (h_1 - h_\infty)\rho^t] \\ &= h_\infty - \rho^{T-1}(h_1 - h_\infty) + \frac{(h_1 - 2h_\infty)}{T} \left(\frac{1 - \rho^T}{1 - \rho} \right). \end{aligned} \quad (7)$$

The term $\tilde{h}$ is a weighted average of the initial and long-run variances. Also, it is the discrete-time approximation of the integral proposed by Lewis (2000) in his volatility-of-volatility (vol-of-vol) series expansion for the option price (Rouah, 2013).

⁴This reduces the variance of h_t process to $4\alpha^2\gamma^2h_t$. A similar result is obtained for the AGARCH model, implying that the latter also represents an SV model in discrete-time.

⁵Choi (2005) applied a similar approximation to the AGARCH model, concluding that the total returns are normally distributed in that case. He also found that at-the-money options priced under this assumption are broadly in line with simulated values.

2.3 Mazzoni's results

We note that recursion implied by (4) and (5) is non-linear in B_t . Mazzoni (2010) uses linear approximations for the non-linear functions: (i) $\log(1 - 2\alpha B_{t+1})$ and (ii) $1/(1 - 2\alpha B_{t+1})$, deriving the following closed-form expression for the log-MGF of X_T :

$$m_X(\varphi) = \varphi \delta T + \frac{\varphi(\varphi + 2\lambda)}{2} \left\{ T \left(\frac{\omega + \alpha}{1 - \phi_\varphi} \right) + \left[h_1 - \left(\frac{\omega + \alpha}{1 - \phi_\varphi} \right) \right] \left(\frac{1 - \phi_\varphi^T}{1 - \phi_\varphi} \right) \right\}, \quad (8)$$

where $\phi_\varphi = \alpha\gamma^2 + \beta - 2\alpha\gamma\varphi + \alpha\varphi^2$. Based on (8), he reports the first four cumulants (κ_1 , κ_2 , κ_3 , and κ_4), under the risk-neutral measure (see Appendix A).

2.4 Log-MGF approximation

We propose a simplification of Mazzoni's (2010) result which involves using second-order approximations of the non-linear terms of α presented in (8).⁶ This gives us the following fourth order polynomial log-MGF (see Appendix C):

$$\begin{aligned} m_X(\varphi) = & \varphi (\delta + \lambda \bar{h}) T + \frac{1}{2} \varphi^2 \left(\bar{h} - \frac{4\alpha\gamma\lambda}{1-\rho} \tilde{h} \right) T \\ & + \frac{1}{6} \varphi^3 \left(\frac{6\alpha\gamma}{1-\rho} \right) \left[\left(\frac{\lambda}{\gamma} - 1 \right) \tilde{h} + \frac{4\alpha\gamma\lambda}{1-\rho} \hat{h} \right] T \\ & + \frac{1}{24} \varphi^4 \left(\frac{12\alpha}{1-\rho} \right) \left[\tilde{h} + \left(\frac{\gamma}{\lambda} - 2 \right) \frac{4\alpha\gamma\lambda}{1-\rho} \hat{h} \right] T + O(\alpha^3). \end{aligned} \quad (9)$$

where $\hat{h} = h_\infty - \rho^{T-1}(h_1 - 2h_\infty) + (h_1 - 3h_\infty)\Sigma_T/T - (h_1 - h_\infty)\Gamma_T/T$ is a second-order correction term, with (i) $\Sigma_T = (1 - \rho^T)/(1 - \rho)$ and (ii) $\Gamma_T = 0.5T(T - 1)(1 - \rho)\rho^{T-2}$; meanwhile, $\bar{h}$ and $\tilde{h}$ were defined above.

⁶Compared with (8) the proposed expression explicitly presents the first four moments, without taking the derivatives of the log-MGF.

We note from (9) that cumulant κ_j is related to the $(1/j!)\varphi^j$ term. Thus, the first four cumulants are:⁷

$$\begin{aligned}\kappa_1 &= (\delta + \lambda\bar{h})T, \quad \kappa_2 = \left(\bar{h} - \frac{4\alpha\gamma\lambda}{1-\rho}\tilde{h}\right)T, \quad \kappa_3 = -\frac{6\alpha\gamma}{1-\rho}\left[\left(1 - \frac{\lambda}{\gamma}\right)\tilde{h} - \frac{4\alpha\gamma\lambda}{1-\rho}\hat{h}\right]T, \text{ and} \\ \kappa_4 &= \left(\frac{12\alpha}{1-\rho}\right)\left[\tilde{h} + \left(\frac{\gamma}{\lambda} - 2\right)\frac{4\alpha\gamma\lambda}{1-\rho}\hat{h}\right]T.\end{aligned}$$

An alternative parametrization of the model involves $\lambda = 0$ (without Jensen term adjustment), in which case we have the following expressions for the cumulants:

$$\kappa_1 = \delta T, \quad \kappa_2 = \bar{h}T, \quad \kappa_3 = -\frac{6\alpha\gamma\tilde{h}T}{1-\rho}, \text{ and } \kappa_4 = \left(\frac{12\alpha}{1-\rho}\right)\left(\tilde{h} + \frac{4\alpha\gamma^2}{1-\rho}\hat{h}\right)T.$$

Finally, if variances are time-variant but deterministic ($\alpha = 0$) we have:

$$m_X(\varphi) = \varphi(\delta + \lambda\bar{h})T + \frac{1}{2}\varphi^2\bar{h}T + O(\alpha). \quad (10)$$

Consequently, X_T follows a normal distribution; however, if $\alpha > 0$ and sufficiently small, both κ_3 and κ_4 differ from zero, implying that X_T is not normally distributed.⁸

⁷In Appendix A we show that these cumulants are exact to the ones reported in Mazzoni (2010) when the risk-neutral measure is considered, and they can be obtained by imposing the following constraints: (i) $\delta = f$ (the drift coefficient is the risk free rate), and (ii) $\lambda = -1/2$ (Jensen term adjustment). This implies $\kappa_1 = (f - 0.5\bar{h})T$, $\kappa_2 = (\bar{h} + \theta\tilde{h})T$, and $\kappa_3 = -3\theta(\pi\tilde{h} + \theta\hat{h})T$ with $\theta = 2\alpha\gamma/(1-\rho)$ and $\pi = (1 + 0.5/\gamma)$. In contrast, the fourth cumulant is an approximation of his result given that it also depends on α^3 ; however, if α is small then any terms of order α^3 or higher should be negligible, providing an accurate estimate of the cumulant. We confirm this in Appendix A.3, where we present the replication of the Mazzoni's (2010) simulations.

⁸In practice, many confounding effects could affect the results. For instance, if in addition to small α , the persistence is close to one, then both $\tilde{h}$ and $\hat{h}$ are close to zero; in this case, κ_3 and κ_4 are small, implying that Gaussian approximation is appropriate.

2.5 AGARCH model

A major limitation of the HN model is its absence from standard statistical software packages, which has consequently constrained the extent of available empirical evidence. A relatively similar model is the Asymmetric GARCH (AGARCH), which is defined by the following equations:⁹

$$r_t = \delta_* + \sqrt{h_t}v_t, \quad (11)$$

$$h_{t+1} = \omega_* + \alpha_* \left(\sqrt{h_t}v_t - \gamma_* \right)^2 + \beta_* h_t, \quad (12)$$

where v_t is a standard Gaussian error term, and parameter restrictions are: $\alpha_* > 0$, $\beta_* > 0$, $\alpha_* + \beta_* < 1$, and $\omega_* + \alpha_*\gamma_*^2 > 0$. Furthermore, the persistence of h_t is defined as follows $\rho_* = \alpha_* + \beta_*$, meanwhile the long-run variance is $h_\infty = (\omega_* + \alpha_*\gamma_*^2)/(1 - \rho_*)$.¹⁰

Our aim is to match the conditional variance sequence of the HN model ($\lambda = 0$) with the one inferred from the AGARCH model. We first note that $(\sqrt{h_t}v_t - \gamma_*)^2 = \gamma_*^2[1 - (1/\gamma_*)\sqrt{h_t}v_t]^2$, which is equal to $\gamma_*^2[1 - 2(1/\gamma_*)\sqrt{h_t}v_t + (1/\gamma_*)^2h_tv_t^2]$, but can be expanded as follows $\gamma_*^2[v_t^2 - 2(1/\gamma_*)\sqrt{h_t}v_t + (1/\gamma_*)^2h_t - (v_t^2 - 1) + (1/\gamma_*)^2h_t(v_t^2 - 1)]$. We assume that $v_t^2 \approx 1$, implying that the last two terms in the previous expression are null.

Thus, our proposal is to consider $(\sqrt{h_t}v_t - \gamma_*)^2 \approx \gamma_*^2[v_t - (1/\gamma_*)\sqrt{h_t}]^2$, which implies the following relationships between the parameters of HN and AGARCH models: (i) $\omega = \omega_*$, (ii) $\alpha = \alpha_*\gamma_*^2$, (iii) $\gamma = 1/\gamma_*$, and (iv) $\beta = \beta_*$.¹¹

⁹Engle (1990) introduced it as a part of commentary and later was included in a list of ARCH/GARCH models (Engle and Ng, 1993).

¹⁰The covariance between returns and variances is $\text{Cov}_{t-1}(h_{t+1}, r_t) = -2\alpha_*\gamma_*h_t$, and the volatility of the variance is $\text{Vol}_{t-1}(h_{t+1}) = \sqrt{2h_t^2 + 4\gamma_*^2h_t\alpha_*}$.

¹¹It is straightforward to confirm that this calibration ensures that both models preserve the same persistence, long-run variance, and covariance between variances and returns; however, as we discuss before, the assumption $v_t^2 \approx 1$ reduces the variance of h_t .

3 Monte Carlo Experiments

In this section, we evaluate the approximations presented above. First, we discuss the parameters of both HN and AGARCH models, to define relevant scenarios. Second, in line with Mazzoni (2010), we evaluate the accuracy of the closed-form cumulants for the HN model and extend the analysis to the AGARCH case, using the calibration outlined above.

3.1 Parameter calibration

A number of related studies provide empirical estimates of the HN model. First, the results in Christoffersen et al. (2006) and Christoffersen et al. (2013) are based on 13 and 21 years of daily S&P 500 returns, respectively. These are: $3.7e-6$ and $3.4e-6$ for α , 0.90 and 0.84 for β , and 96% and 97% for $\rho = \alpha\gamma^2 + \beta$. Thus, most of the persistence component (ρ) is explained by β . Second, Mazzoni (2010) reports results for three stock market indices: S&P 500, Dow Jones, and DAX, using 20 years of daily data. Estimates for ω , α , and γ are aligned with the level of the long-run volatility. For instance, results for α are $3.4e-6$, $3.6e-6$, and $5.3e-6$, respectively for each market. Estimates for β are 0.85, 0.85, and 0.88; meanwhile, overall persistence estimates are around 97%. Finally, Mozumder et al. (2024) propose an alternative estimate of the HN model by considering the S&P 500 options' volatility smile in years 2016, 2017, and 2018. Estimates for α are $1.5e-7$, $3.16e-7$, and $9.97e-7$, while estimates for β are 0.98, 0.94, and 0.92; and the persistence coefficient estimates are 99.8%, 98.9% and 98%.

Drawing on the empirical findings we discuss above, and imposing $\delta = \lambda = 0$, we consider two scenarios to simulate the HN model: (i) Low-LRV with $\omega = 1e-8$, $\alpha = 3.6e-6$, $\beta = 0.85$, and $\gamma = 188$, which implies $\rho = 0.977$ and an annual (252 days) LRV of 20%,

and (ii) High-LRV with $\omega = 2e - 7$, $\alpha = 3.6e - 6$, $\beta = 0.85$, and $\gamma = 200$, which implies $\rho = 0.994$ and an annual LRV of 40%.

Furthermore, Table 1 summarizes estimates of the AGARCH model for US stock indices, based on V-Lab website.¹² The less risky ones (S&P 500 and Dow-Jones) display negative estimates for ω_* , however, the average estimate across the stock indices is 0.003, while all other parameters are similar across indices: α_* around 10%, β_* around 88%, and γ_* around 60%. Given that the parameter estimates are obtained from returns expressed in percentage terms (i.e. multiplied by 100), ω_* should be divided by 10,000 and γ_* should be divided by 100. Then, our calibration for the HN model is: $\omega = 3e - 7$, $\alpha = 0.1 \times 0.006^2 = 3.6e - 6$, $\beta = 0.88$, $\gamma = 1/0.006 \approx 166.67$. The resulting calibration is broadly aligned with the empirical evidence discussed earlier, lending support to the strategy of calibrating the HN model parameters using the estimates obtained from the AGARCH specification.

Table 1: AGARCH estimates for US Stock Indices

	S&P 500	Dow Jones	Nasdaq	S&P MidCap	S&P SmallCap	Russell 2000
ω_*	-0.0053 (0.0037)	-0.0059 (0.0017)	0.0141 (0.0021)	0.0015 (0.0015)	0.0077 (0.0013)	0.0062 (0.0011)
α_*	0.1028 (0.0036)	0.0957 (0.0023)	0.1018 (0.0021)	0.0948 (0.0019)	0.0935 (0.0018)	0.0995 (0.0018)
β_*	0.8684 (0.0038)	0.8724 (0.0025)	0.8768 (0.0022)	0.8792 (0.0021)	0.8851 (0.0019)	0.8842 (0.0019)
γ_*	0.6306 (0.0377)	0.6536 (0.0180)	0.5430 (0.0243)	0.6224 (0.0150)	0.5568 (0.0133)	0.5091 (0.0122)
ρ_*	0.9712	0.9681	0.9786	0.9740	0.9786	0.9837
h_∞	1.2354	1.0966	2.0615	1.4701	1.7144	1.9625
LRV	17.6	16.6	22.8	19.2	20.8	22.2

This table reports estimates from the AGARCH model outlined in Section 2.5. Standard errors are in parentheses. The sample period extends from Jan. 2, 1990 to Feb. 13, 2026. Source: V-Lab.

¹²V-Lab (Volatility Laboratory) is a hub of researchers and data scientists associated with NYU Stern. In the website: <https://vlab.stern.nyu.edu/>, several tools are freely available, including estimates of GARCH models for many financial assets.

3.2 Moments Accuracy

Figure 1 shows the analytical moments and Monte Carlo simulations using the parameters of the HN model for the first scenario (Low-LRV), and two alternative initial variances. In line with the results presented in Mazzoni (2010), the precision of the analytical moments is very high; regardless of the initial variance. However, the analytical expression for the kurtosis falls towards the lower end of the simulated range. This is an anticipated result, since the linearization of the recursion relies on the assumption that α is small. Figure 2 expands the previous results to the second scenario (High-LRV), when the LRV is significantly higher. Results are consistent with the previous case. In light of this evidence, we confirm that our analytical expressions can be used to compute moments associated to cumulative returns generated by a HN model. This aligns with Mazzoni’s (2010) main results, although we used a slightly different approach to obtain analytical expressions for the first four cumulants.

Figure 3 reports simulated returns from the AGARCH model and contrasts them with the analytical moments of the HN model obtained under our calibration. The findings demonstrate alignment in variance and skewness, although this does not extend to the kurtosis. As discussed in the preceding section, the models differ with respect to the volatility of the conditional variance. Consequently, discrepancies in kurtosis are expected. Furthermore, our parameter calibration is based on the assumption $v_t^2 \approx 1$, which implies a reduction in the variance of h_t . This reduction mainly influences the kurtosis, while leaving the persistence, long-run variance, and covariance between variances and returns consistent with the empirical calibration.¹³

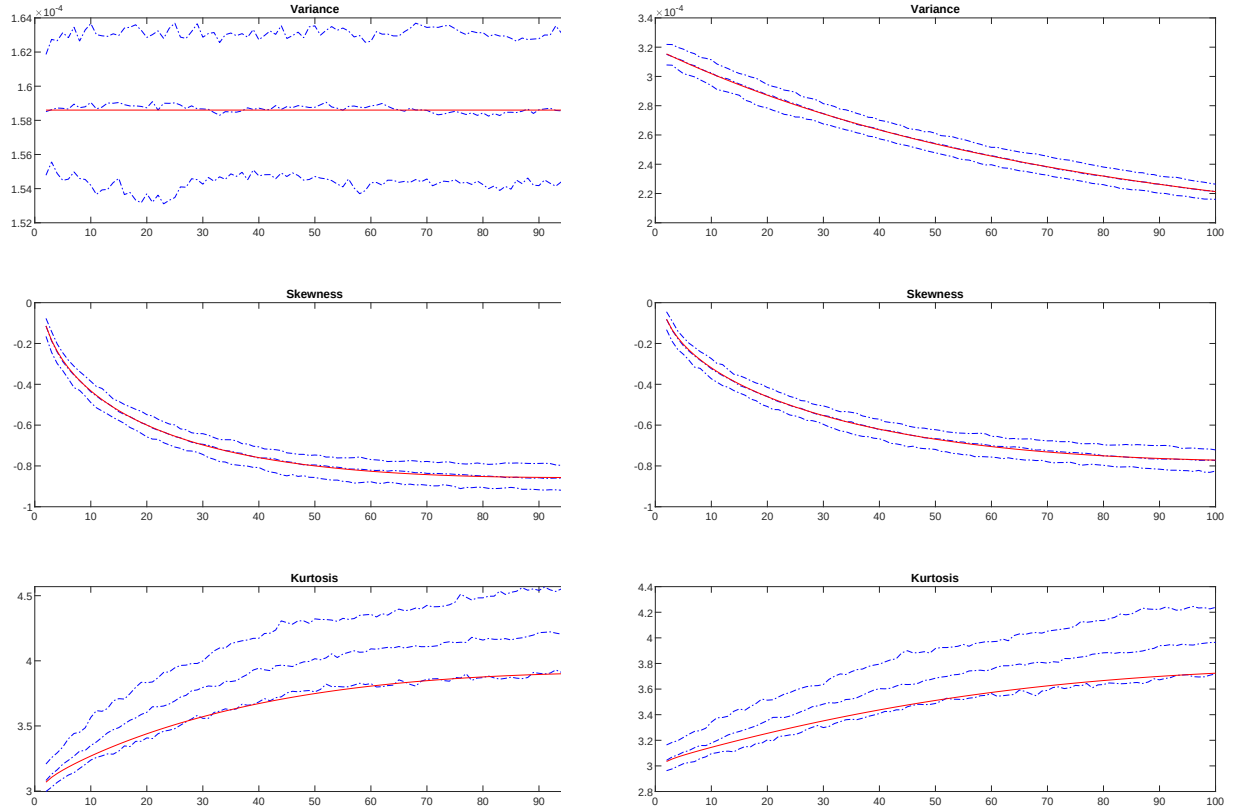
¹³The discrepancy in the kurtosis prevents the use of parameter calibration for tail-indicators (e.g., Value at Risk); however, a complete assessment of that is beyond the scope of this paper.

4 Conclusion

Mazzoni(2010) shows that by linearizing the recursion system of the Heston–Nandi model, one obtains a closed-form expression for the log-MGF of cumulative returns (total returns). We further extend this result by approximating the expression with a fourth order polynomial which directly relates the cumulants or moments of total returns. Based on that, we provide analytic expressions for the first four moments and confirm their accuracy with Monte Carlo simulations. Our analytical expressions combine different averages of the initial and long-run variances ($\bar{h}$ and $\tilde{h}$), being consistent with the literature related to the vol-of-vol expansion of the Heston’s (1993) model.

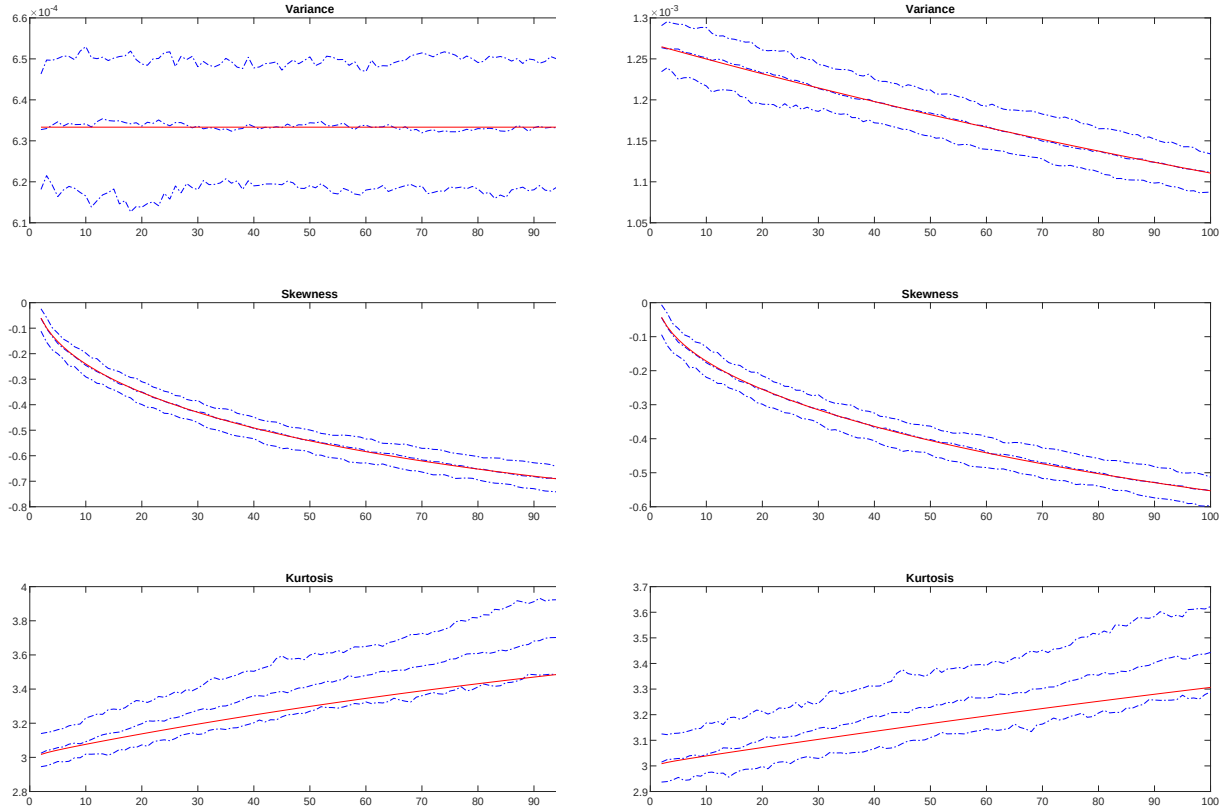
We show that the first three moments of the AGARCH model can be approximated using the same expressions obtained under a simple calibration. We acknowledge that the analytical expression for kurtosis performs less accurately, consistent with the substantial differences in the volatility of variance across the two models. Nevertheless, there remains clear value in using analytical expressions for targeted applications, including the pricing of at-the-money and risk-reversal contracts. We leave the implementation of these applications to future research.

Figure 1: Heston Nandi model moments based on scenario Low-LRV

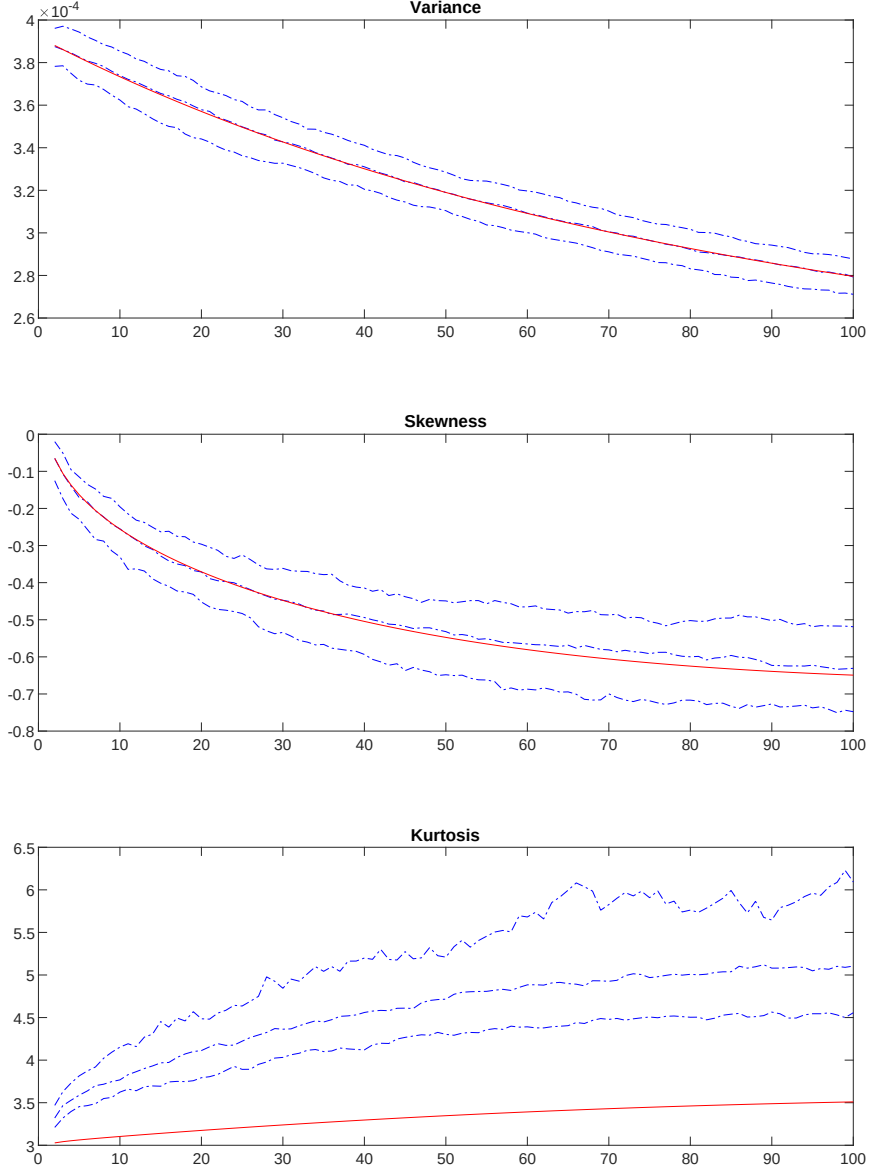


The figure presents the moments (variance, skewness, and kurtosis) implied by the Heston–Nandi model calibrated using the parameter set for scenario Low-LRV presented in Section 3.1. The solid red line corresponds to the Heston–Nandi analytical cumulants introduced in Section 2.4. The dashed blue lines represent the 5th, 50th, and 95th empirical quantiles obtained from a Monte Carlo (MC) simulation exercise. These quantiles are derived from an MC simulation that generates 1,000,000 log-return paths from the Heston–Nandi model and groups them into 100 equally sized blocks to compute empirical moments. Panel (a) assumes $h_1 = h_\infty$, whereas Panel (b) assumes $h_1 = 2h_\infty$. Source: Authors' calculations.

Figure 2: Heston Nandi model moments based on scenario High-LRV



The figure presents the moments (variance, skewness, and kurtosis) implied by the Heston-Nandi model calibrated using the parameter set for scenario High-LRV presented in Section 3.1. The solid red line corresponds to the Heston-Nandi analytical cumulants introduced in Section 2.4. The dashed blue lines represent the 5th, 50th, and 95th empirical quantiles obtained from a Monte Carlo (MC) simulation exercise. These quantiles are derived from an MC simulation that generates 1,000,000 log-return paths from the Heston-Nandi model and groups them into 100 equally sized blocks to compute empirical moments. Panel (a) assumes $h_1 = h_\infty$, whereas Panel (b) assumes $h_1 = 2h_\infty$. Source: Authors' calculations.



The figure presents the empirical moments (variance, skewness, and kurtosis) of log-returns. The dashed blue lines represent the 5th, 50th, and 95th empirical quantiles obtained from a Monte Carlo (MC) simulation exercise based on the AGARCH model introduced in Section 2.5 and calibrated with the parameters described in Section 3.1, assuming that $h_1 = 2h_\infty$. These quantiles are derived from an MC simulation that generates 1,000,000 log-return paths from the AGARCH model and groups them into 100 equally sized blocks to compute empirical moments. The solid red line represents the Heston–Nandi analytical cumulants introduced in Section 2.4. Source: Authors’ calculations.

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A Mazzoni's result

Heston and Nandi (2000) provide the following recursion equations to obtain the conditional moment generating function:

$$A_t = A_{t+1} + \varphi\delta + \omega B_{t+1} - \frac{1}{2} \log(1 - 2\alpha B_{t+1}), \quad (\text{A.1})$$

$$B_t = \varphi(\gamma + \lambda) - \frac{\gamma^2}{2} + \beta B_{t+1} + \frac{0.5(\varphi - \gamma)^2}{1 - 2\alpha B_{t+1}}, \quad (\text{A.2})$$

and $A_T = B_T = 0$ are the terminal conditions.

Clearly, there is not a closed-form expression for the solution to the non-linear recursion; however, Mazzoni (2010) suggests to approximate the non-linear terms in (A.1) and (A.2) with linear functions. After that, he derives a closed-form expression for the logarithm of the Moment Generating Function (log-MGF).¹⁴

A.1 Approximation of the log-MGF

First, we consider the following approximation $(1 - 2\alpha B_{t+1})^{-1} \approx 1 + 2\alpha B_{t+1}$, which is valid for small value of α . Replacing this in equation (A.2) and canceling terms, we have:

$$B_t = \mu_\varphi + \phi_\varphi B_{t+1} \text{ with } \mu_\varphi = 0.5\varphi(\varphi + 2\lambda) \text{ and } \phi_\varphi = \alpha\gamma^2 + \beta - 2\alpha\gamma\varphi + \alpha\varphi^2.$$

Using terminal condition $B_T = 0$ we could solve the previous equation backward: $B_{T-1} = \mu_\varphi$, $B_{T-2} = \mu_\varphi + \phi_\varphi B_{T-1} = \mu_\varphi(1 + \phi_\varphi)$, and $B_{T-3} = \mu_\varphi + \phi_\varphi B_{T-2} = \mu_\varphi(1 + \phi_\varphi + \phi_\varphi^2)$. Thus, for $J \geq 1$, the solution is:

$$B_{T-J}(\varphi) = \mu_\varphi \left(\frac{1 - \phi_\varphi^J}{1 - \phi_\varphi} \right). \quad (\text{A.3})$$

¹⁴This section follows Mazzoni (2010).

Second, we consider that $\log(1 - 2\alpha B_{t+1}) \approx -2\alpha B_{t+1}$, which is valid for small value of α .

Replacing this in equation (A.1) we have: $A_t = A_{t+1} + \varphi\delta + (\omega + \alpha)B_{t+1}$.

Using terminal conditions $A_T = 0$ and $B_T = 0$, we could solve the previous equation backward: $A_{T-1} = \varphi\delta$, $A_{T-2} = A_{T-1} + \varphi\delta + (\omega + \alpha)B_{T-1} = 2\varphi\delta + (\omega + \alpha)B_{T-1}$, and $A_{T-3} = A_{T-2} + \varphi\delta + (\omega + \alpha)B_{T-2} = 3\varphi\delta + (\omega + \alpha)(B_{T-1} + B_{T-2})$. Thus, for $J \geq 1$, we can express $A_{T-J} = J\varphi\delta + (\omega + \alpha)\sum_{j=0}^{J-1} B_{T-j}$. Using (A.3) and solving the sum, we have the following expression for A_{T-J} :

$$A_{T-J}(\varphi) = \varphi\delta J + \mu_\varphi \left(\frac{\omega + \alpha}{1 - \phi_\varphi} \right) \left[J - \left(\frac{1 - \phi_\varphi^J}{1 - \phi_\varphi} \right) \right]. \quad (\text{A.4})$$

Finally, the log-MGF of the total returns (X_T) is: $m_X(\varphi) = A_0(\varphi) + h_1 B_0(\varphi)$, where h_1 is an initial given variance; meanwhile, $A_0(\varphi)$ and $B_0(\varphi)$ are obtained by setting $J = T$ in (A.3) and (A.4), which implies the following expressions:

$$B_0(\varphi) = \mu_\varphi \left(\frac{1 - \phi_\varphi^T}{1 - \phi_\varphi} \right) \text{ and } A_0(\varphi) = \varphi\delta T + \mu_\varphi \left(\frac{\omega + \alpha}{1 - \phi_\varphi} \right) \left[T - \left(\frac{1 - \phi_\varphi^T}{1 - \phi_\varphi} \right) \right].$$

Reordering these terms we could express the log-MGF as:

$$m_X(\varphi) = \varphi\delta T + \mu_\varphi \left\{ T \left(\frac{\omega + \alpha}{1 - \phi_\varphi} \right) + \left[h_1 - \left(\frac{\omega + \alpha}{1 - \phi_\varphi} \right) \right] \left(\frac{1 - \phi_\varphi^T}{1 - \phi_\varphi} \right) \right\}. \quad (\text{A.5})$$

A.2 Cumulants

Cumulants can be obtained by taking derivatives of the log-MGF. Mazzoni (2010) reports the first four cumulants in the Appendix of his paper, under the risk-neutral measure. This implies the following parametrization of the model: (i) $\delta = f$, where f is the risk-free rate,

and (ii) $\lambda = -0.5$, needed for the Jensen adjustment.

Moreover, three of the cumulants could be written more compactly using both the definition of the long-run variance, which is $h_\infty = (\omega + \alpha)/(1 - \rho)$, and $\Sigma_T = (1 - \rho^T)/(1 - \rho)$. From his appendix, the first cumulant (κ_1) does not depend on α , being:

$$\kappa_1 = fT - \frac{h_1}{2}\Sigma_T + \frac{1}{2}(\Sigma_T - T)h_\infty = \left(f - \frac{1}{2}\bar{h}\right)T, \quad (\text{A.6})$$

where $\bar{h} = h_\infty + (h_1 - h_\infty)\Sigma_T/T$ is the average of the deterministic time-variant variance.

The second cumulant (κ_2) is linear in α , and it is presented as:

$$\begin{aligned} \kappa_2 = & h_1\Sigma_T - \frac{2\alpha}{1-\rho}(\Sigma_T - T)h_\infty\gamma - h_1\left(\frac{2\alpha\gamma}{1-\rho}\right)(T\rho^{T-1} - \Sigma_T) \\ & + h_\infty\left(T - \Sigma_T + \frac{2T\alpha\rho^{T-1}\gamma}{1-\rho} - \frac{2\alpha\gamma}{1-\rho}\Sigma_T\right), \end{aligned} \quad (\text{A.7})$$

which could be written as follows:

$$\begin{aligned} \kappa_2 = & h_1\Sigma_T - \left(\frac{2\alpha\gamma}{1-\rho}\right)\Sigma_T h_\infty + \left(\frac{2\alpha\gamma}{1-\rho}\right)Th_\infty - \left(\frac{2\alpha\gamma}{1-\rho}\right)T\rho^{T-1}h_1 \\ & + \left(\frac{2\alpha\gamma}{1-\rho}\right)\Sigma_T h_1 + h_\infty T - h_\infty\Sigma_T + \left(\frac{2\alpha\gamma}{1-\rho}\right)T\rho^{T-1}h_\infty - \left(\frac{2\alpha\gamma}{1-\rho}\right)\Sigma_T h_\infty \\ = & h_\infty T + (h_1 - h_\infty)\Sigma_T + \left(\frac{2\alpha\gamma}{1-\rho}\right)[Th_\infty + \Sigma_T h_1 - 2\Sigma_T h_\infty - T\rho^{T-1}h_1 + T\rho^{T-1}h_\infty] \\ = & \left[h_\infty + \frac{(h_1 - h_\infty)}{T}\Sigma_T\right]T + \left(\frac{2\alpha\gamma}{1-\rho}\right)\left[h_\infty - \rho^{T-1}(h_1 - h_\infty) + \frac{(h_1 - 2h_\infty)}{T}\Sigma_T\right]T. \end{aligned}$$

The first bracket is $\bar{h}$, presented above. Defining $\tilde{h} = h_\infty - \rho^{T-1}(h_1 - h_\infty) + (h_1 - 2h_\infty)\Sigma_T/T$, we have the following compact expression for the second cumulant:

$$\kappa_2 = (\bar{h} + \theta\tilde{h})T, \text{ with } \theta = \frac{2\alpha\gamma}{1-\rho}. \quad (\text{A.8})$$

The third cumulant, reported by the author, includes terms that are proportional to both α and α^2 . Noting that if $\alpha = 0$ then total returns are symmetry, which is in line with normal distribution. We could write the cumulant using the following auxiliary terms:

$$\begin{aligned} H_1 &= \frac{2\alpha\gamma T\rho^{T-1}}{1-\rho} - \frac{2\alpha\gamma(1-\rho^T)}{(1-\rho)^2} = \frac{2\alpha\gamma}{1-\rho} (T\rho^{T-1} - \Sigma_T), \\ H_2 &= \frac{2\alpha}{(1-\rho)^2} + \frac{8\alpha^2\gamma^2}{(1-\rho)^3} = \frac{2\alpha}{(1-\rho)^2} \left(1 + \frac{4\alpha\gamma^2}{1-\rho}\right), \text{ and} \\ H_3 &= -\frac{2\alpha T\rho^{T-1} + 4\alpha^2\gamma^2 T(T-1)\rho^{T-2}}{1-\rho} - \frac{8\alpha^2\gamma^2 T\rho^{T-1}}{(1-\rho)^2}. \end{aligned}$$

Based on these terms, we could express the third cumulant as:

$$\begin{aligned} \kappa_3 &= 3h_1 H_1 - \frac{6\alpha(\omega + \alpha)\gamma}{(1-\rho)^2} (T - \Sigma_T + H_1) + \frac{3}{2}(\omega + \alpha)(\Sigma_T - T)H_2 \\ &\quad - \frac{3}{2}h_1 [(1-\rho^T)H_2 + H_3] + \frac{\omega + \alpha}{1-\rho} \left\{ -3H_1 + \frac{3}{2} [(1-\rho^T)H_2 + H_3] \right\}. \end{aligned} \quad (\text{A.9})$$

Replacing H_1 , H_2 and H_3 , and collecting terms, we have:

$$\begin{aligned} \kappa_3 &= \left(\frac{6\alpha\gamma}{1-\rho} + \frac{3\alpha}{1-\rho} \right) [(h_1 - h_\infty)(T\rho^{T-1} - \Sigma_T) + h_\infty(\Sigma_T - T)] \\ &\quad + 12 \left(\frac{\alpha\gamma}{1-\rho} \right)^2 [(h_1 - 2h_\infty)(T\rho^{T-1} - \Sigma_T) + h_\infty(\Sigma_T - T)] \\ &\quad + 6 \left(\frac{\alpha\gamma}{1-\rho} \right)^2 (h_1 - h_\infty)T(T-1)(1-\rho)\rho^{T-2}. \end{aligned}$$

Considering the term $\hat{h} = h_\infty - \rho^{T-1}(h_1 - 2h_\infty) + (h_1 - 3h_\infty)\Sigma_T/T - (h_1 - h_\infty)\Gamma_T/T$, with $\Gamma = 0.5T(T-1)(1-\rho)\rho^{T-2}$, and $\tilde{h}$ as defined above, then κ_3 could be expressed more

compactly as follows:

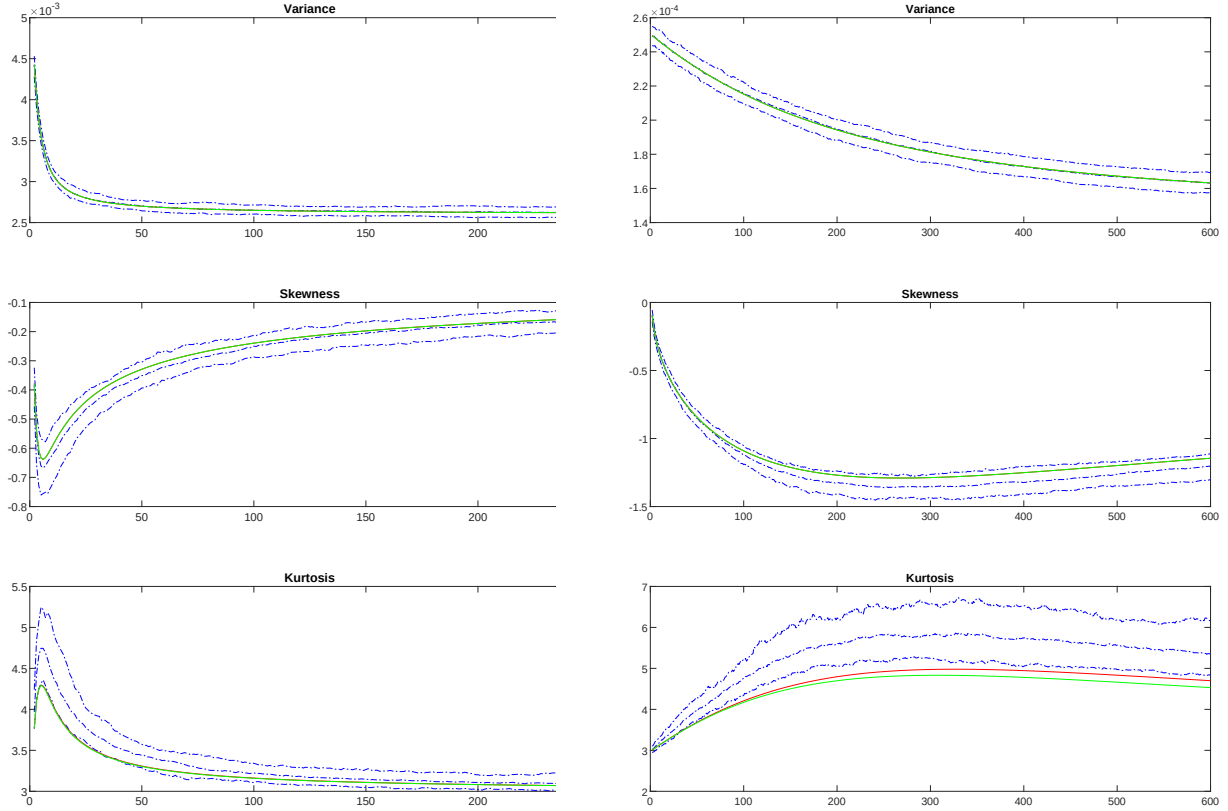
$$\begin{aligned}
\kappa_3 &= -3(1+2\gamma)\frac{\alpha}{1-\rho}\tilde{h}T - 12\left(\frac{\alpha\gamma}{1-\rho}\right)^2\hat{h}T \\
&= -\frac{6\alpha\gamma}{1-\rho}\left[\left(1+\frac{1}{2\gamma}\right)\tilde{h} + \frac{2\alpha\gamma}{1-\rho}\hat{h}\right]T \\
&= -3\theta\left[\left(1+\frac{1}{2\gamma}\right)\tilde{h} + \theta\hat{h}\right]T.
\end{aligned} \tag{A.10}$$

Finally, the fourth cumulant is a lengthy expression that includes terms proportional to α , α^2 , and α^3 ; therefore, it will not be comparable with the expression derived in this paper.

A.3 Simulation results

As we discuss in the text, our analytical expressions for the first three cumulants are exact with respect to the one presented by Mazzoni (2010). Our expression for the kurtosis is quadratic in α , while the one proposed by the author is cubic. We believe that difference should be negligible as α is assumed to be small. To evaluate this, we replicate the extreme scenarios discussed in Mazzoni (2010), noting that our expression for the kurtosis does not differ significantly to one presented by the author.

Figure 4: Mazzoni (2010) comparison



The figure presents the moments (variance, skewness, and kurtosis) implied by the Heston–Nandi model calibrated using the parameter in Mazzoni (2010). Panel (a) and (b) presents exhibits 3 and 6 as in Mazzoni (2010), respectively. The solid red line represents the Heston–Nandi analytical cumulants introduced in Mazzoni (2010). The solid green line corresponds to the Heston–Nandi analytical cumulants introduced in Section 2.4. The dashed blue lines represent the 5th, 50th, and 95th empirical quantiles obtained from a Monte Carlo (MC) simulation exercise. These quantiles are derived from an MC simulation that generates 1,000,000 log-return paths from the Heston–Nandi model and groups them into 100 equally sized blocks to compute empirical moments. Source: Authors' calculations.

B Gaussian Approximation

In order to obtain a Gaussian approximation of X_T , we consider a perturbation approximation of the cumulative variances.

For instance, when $T = 3$ we have: h_1 as given, $h_2 = h_\infty + (h_1 - h_\infty)\rho + \eta\sqrt{h_1}u_1$, and $h_3 = h_\infty + (h_1 - h_\infty)\rho^2 + \eta(\sqrt{h_2}u_2 + \rho\sqrt{h_1}u_1)$. Thus, the sum of h 's could be expressed as: $h_1 + h_2 + h_3 = 3h_\infty + (h_1 - h_\infty)(1 + \rho + \rho^2) + \eta[(1 + \rho)\sqrt{h_1}u_1 + \sqrt{h_2}u_2]$.

Noting that $(1 + \rho + \rho^2) = (1 - \rho^3)/(1 - \rho)$ and $(1 + \rho) = (1 - \rho^2)/(1 - \rho)$, we have the following expression for this small sample case:

$$\sum_{t=0}^2 h_{t+1} = \left[h_\infty + \frac{(h_1 - h_\infty)}{3} \left(\frac{1 - \rho^3}{1 - \rho} \right) \right] 3 + \eta \sum_{t=0}^2 \left(\frac{1 - \rho^{2-t}}{1 - \rho} \right) \sqrt{h_{t+1}} u_{t+1}. \quad (\text{B.1})$$

A generalization of the previous sum for a given sample size of T is:

$$\sum_{t=0}^{T-1} h_{t+1} = \left[h_\infty + \frac{(h_1 - h_\infty)}{T} \left(\frac{1 - \rho^T}{1 - \rho} \right) \right] T + \eta \sum_{t=0}^{T-1} \left(\frac{1 - \rho^{(T-1)-t}}{1 - \rho} \right) \sqrt{h_{t+1}} u_{t+1}. \quad (\text{B.2})$$

The first bracket above is $\bar{h}$, as defined before; meanwhile, the second term is stochastic and can be approximated by considering that $\sqrt{h_{t+1}} \approx \sqrt{h_\infty + (h_1 - h_\infty)\rho^t}$, which is valid if η is small. Thus, total returns could be expressed as:

$$\begin{aligned} X_T &= \delta T + \lambda \sum_{t=0}^{T-1} h_{t+1} + \sum_{t=0}^{T-1} \sqrt{h_{t+1}} u_{t+1} \\ &\approx (\delta + \lambda \bar{h})T + \sum_{t=0}^{T-1} \left[1 + \lambda \eta \left(\frac{1 - \rho^{(T-1)-t}}{1 - \rho} \right) \right] \sqrt{h_\infty + (h_1 - h_\infty)\rho^t} u_{t+1}. \end{aligned} \quad (\text{B.3})$$

Given that the last sum is an average of Gaussian disturbances, then the distribution of X_T is approximately Gaussian when η is small.

The variance of X_T could be obtained by replacing the Gaussian disturbances with a one disturbance with a proper variance, which is obtained as follows:

$$\begin{aligned}
\text{Var}(X_T) &= \sum_{t=0}^{T-1} \left[1 + \lambda\eta \left(\frac{1 - \rho^{(T-1)-t}}{1 - \rho} \right) \right]^2 [h_\infty + (h_1 - h_\infty)\rho^t] \\
&= \sum_{t=0}^{T-1} \left[1 + 2\lambda\eta \left(\frac{1 - \rho^{(T-1)-t}}{1 - \rho} \right) \right] [h_\infty + (h_1 - h_\infty)\rho^t] + O(\eta^2) \\
&= \bar{h}T + \frac{2\lambda\eta}{1 - \rho} \sum_{t=0}^{T-1} [1 - \rho^{(T-1)-t}] [h_\infty + (h_1 - h_\infty)\rho^t] + O(\eta^2).
\end{aligned} \tag{B.4}$$

where $O(\eta^2)$ stands for term of order η^2 or higher.

Following the discussion presented in the main text, we consider the following terms:

$$\begin{aligned}
\tilde{h} &= \frac{1}{T} \sum_{t=0}^{T-1} [1 - \rho^{(T-1)-t}] [h_\infty + (h_1 - h_\infty)\rho^t] \\
&= h_\infty - \rho^{T-1}(h_1 - h_\infty) + \frac{(h_1 - 2h_\infty)}{T} \left(\frac{1 - \rho^T}{1 - \rho} \right),
\end{aligned} \tag{B.5}$$

and $\vartheta = 2\lambda\eta/(1 - \rho)$. Based on that we have $\text{Var}(X_T) = (\bar{h} + \vartheta\tilde{h})T$, which support that X_T can be approximated with a Gaussian distribution when η is small.

C Log-MGF Second-order Approximation

Let $\rho = \alpha\gamma^2 + \beta$ be the persistence of the conditional variance in equation (2), therefore we could write $\phi_\varphi = \rho + (\varphi^2 - 2\gamma\varphi)\alpha$.

Given that we will consider second order approximations of the nonlinear expressions that are functions of $\phi_\varphi = \phi_\varphi(\alpha)$.

First, for the expression $1/(1 - \phi_\varphi)$ we have:

$$\begin{aligned} \frac{1}{1 - \phi_\varphi} &= \frac{1}{1 - \rho} + \frac{1}{(1 - \rho)^2}(\varphi^2 - 2\gamma\varphi)\alpha + \frac{1}{(1 - \rho)^3}(\varphi^2 - 2\gamma\varphi)^2\alpha^2 + O(\alpha^3) \quad (\text{C.1}) \\ &= \left(\frac{1}{1 - \rho} \right) (1 + \psi_\varphi\alpha + \psi_\varphi^2\alpha^2) + O(\alpha^3), \end{aligned}$$

where $\psi_\varphi = (\varphi^2 - 2\gamma\varphi)/(1 - \rho)$.

Based on this result we have the following approximations:

$$\frac{\omega + \alpha}{1 - \phi_\varphi} = h_\infty + h_\infty\psi_\varphi\alpha + h_\infty\psi_\varphi^2\alpha^2 + O(\alpha^3), \text{ and} \quad (\text{C.2})$$

$$\left(h_1 - \frac{\omega + \alpha}{1 - \phi_\varphi} \right) = (h_1 - h_\infty) - h_\infty\psi_\varphi\alpha - h_\infty\psi_\varphi^2\alpha^2 + O(\alpha^3). \quad (\text{C.3})$$

Second, considering $\Sigma_T = (1 - \rho^T)/(1 - \rho)$, then for $(1 - \phi_\varphi^T)/(1 - \phi_\varphi)$ we have:

$$\frac{1 - \phi_\varphi^T}{1 - \phi_\varphi} = \Sigma_T + \left[\frac{(1 - \rho^T) + (T\rho^T - T\rho^{T-1})}{(1 - \rho)^2} \right] (\varphi^2 - 2\gamma\varphi)\alpha \quad (\text{C.4})$$

$$+ \left[\Sigma_T - T\rho^{T-1} - \frac{1}{2}T(T-1)(1 - \rho)\rho^{T-2} \right] \frac{(\varphi^2 - 2\gamma\varphi)^2\alpha^2}{(1 - \rho)^2} + O(\alpha^3), \quad (\text{C.5})$$

which implies the following approximation:

$$\frac{1 - \phi_\varphi^T}{1 - \phi_\varphi} = \Sigma_T + (\Sigma_T - T\rho^{T-1})\psi_\varphi\alpha + (\Sigma_T - T\rho^{T-1} - \Gamma_T)\psi_\varphi^2\alpha^2 + O(\alpha^3), \quad (\text{C.6})$$

where $\Gamma_T = 0.5T(T-1)(1-\rho)\rho^{T-2}$.

Using equations (C.2), (C.3), and (C.6) we could obtain the second-order approximation of the log-MGF. In particular, we have:

$$\left(h_1 - \frac{\omega + \alpha}{1 - \phi_\varphi}\right) \left(\frac{1 - \phi_\varphi^T}{1 - \phi_\varphi}\right) = (h_1 - h_\infty)\Sigma_T + \Lambda_{1T}\psi_\varphi\alpha + \Lambda_{2T}\psi_\varphi^2\alpha^2 + O(\alpha^3),$$

where $\Lambda_{1T} = (h_1 - h_\infty)(\Sigma_T - T\rho^{T-1}) - h_\infty\Sigma_T = (h_1 - 2h_\infty)\Sigma_T - (h_1 - h_\infty)T\rho^{T-1}$ and $\Lambda_{2T} = (h_1 - h_\infty)(\Sigma_T - T\rho^{T-1} - \Gamma_T) - h_\infty(\Sigma_T - T\rho^{T-1}) - h_\infty\Sigma_T$.

Also, we note that $T(\omega + \alpha)/(1 - \phi_\varphi) = Th_\infty + Th_\infty\psi_\varphi\alpha + Th_\infty\psi_\varphi^2\alpha^2$, implying that:
 $m_X(\varphi) = \varphi\delta T + \mu_\varphi[Th_\infty + (h_1 - h_\infty)\Sigma_T + (Th_\infty + \Lambda_{1T})\psi_\varphi\alpha + (Th_\infty + \Lambda_{2T})\psi_\varphi^2\alpha^2] + O(\alpha^3)$,
which could be written as follows:

$$m_X(\varphi) = \varphi\delta T + \frac{\varphi(\varphi + 2\lambda)}{2} \left[\bar{h} + \left(\frac{\varphi^2 - 2\gamma\varphi}{1 - \rho} \right) \alpha \tilde{h} + \left(\frac{\varphi^2 - 2\gamma\varphi}{1 - \rho} \right)^2 \alpha^2 \hat{h} \right] T + O(\alpha^3), \quad (\text{C.7})$$

where $\bar{h}$ is the average variance: $\bar{h} = h_\infty + (h_1 - h_\infty)\Sigma_T/T$, and $\tilde{h}$ is the first-order correction term: $\tilde{h} = h_\infty - \rho^{T-1}(h_1 - h_\infty) + (h_1 - 2h_\infty)\Sigma_T/T$, and $\hat{h}$ is the second-order correction term: $\hat{h} = h_\infty - \rho^{T-1}(h_1 - 2h_\infty) + (h_1 - 3h_\infty)\Sigma_T/T - (h_1 - h_\infty)\Gamma_T/T$.

These terms ($\bar{h}$, $\tilde{h}$, and $\hat{h}$) combine the initial variance (h_1) with the long-run one (h_∞). For example, if $h_1 = h_\infty$ (initial variance is at the long-run level), then we have the following: $\bar{h} = h_\infty$, $\tilde{h} = h_\infty(1 - \Sigma_T/T)$, and $\hat{h} = h_\infty[1 + \rho^{T-1} - 2\Sigma_T/T]$. Mazzoni (2010) presents an interesting case in which the initial variance is large ($h_1 = 2h_\infty$). This leads to $\bar{h} = h_\infty(1 + \Sigma_T/T)$, $\tilde{h} = h_\infty(1 - \rho^{T-1})$, and $\hat{h} = h_\infty[1 - \Sigma_T/T - 0.5(T-1)(1-\rho)\rho^{T-2}]$.

Moreover, the persistence parameter (ρ) also plays a role in these terms. For instance, if ρ is close to zero (low-degree of persistence) then $\rho^T \rightarrow 0$ and $\Sigma_T \rightarrow (1 - \rho)^{-1}$. This implies that $\bar{h} \rightarrow h_\infty + (1 - \rho)^{-1}(h_1 - h_\infty)/T$, $\tilde{h} \rightarrow h_\infty + (1 - \rho)^{-1}(h_1 - 2h_\infty)/T$, and $\hat{h} \rightarrow h_\infty + (1 - \rho)^{-1}(h_1 - 3h_\infty)/T$. In contrast, if ρ is close to one (high-degree of persistence as in the Integrated-GARCH) then $\Sigma_T \rightarrow T$, implying that $\bar{h} \rightarrow h_1$, $\tilde{h} \rightarrow 0$, and $\hat{h} \rightarrow 0$. Notably, in this case, X_T could be approximated using a normal distribution.

Cumulants can be obtained from (C.7) by reordering terms in powers of φ . For the first four ones the log-MGF can be expressed as follows:

$$\begin{aligned}
m_X(\varphi) = & \varphi (\delta + \lambda \bar{h}) T + \frac{1}{2} \varphi^2 \left[\bar{h} - \left(\frac{4\alpha\gamma\lambda}{1-\rho} \right) \tilde{h} \right] T \\
& + \frac{1}{6} \varphi^3 \left(\frac{6\alpha\gamma}{1-\rho} \right) \left[\left(\frac{\lambda}{\gamma} - 1 \right) \tilde{h} + \left(\frac{4\alpha\gamma\lambda}{1-\rho} \right) \hat{h} \right] T \\
& + \frac{1}{24} \varphi^4 \left(\frac{12\alpha}{1-\rho} \right) \left[\tilde{h} + \left(\frac{\gamma}{\lambda} - 2 \right) \left(\frac{4\alpha\gamma\lambda}{1-\rho} \right) \hat{h} \right] T + O(\alpha^3).
\end{aligned} \tag{C.8}$$

From previous equation, we have the following expressions for the first four cumulants:

$$\begin{aligned}
\kappa_1 &= (\delta + \lambda \bar{h}) T, \quad \kappa_2 = \left(\bar{h} - \frac{4\alpha\gamma\lambda}{1-\rho} \tilde{h} \right) T, \\
\kappa_3 &= -\frac{6\alpha\gamma}{1-\rho} \left[\left(1 - \frac{\lambda}{\gamma} \right) \tilde{h} - \frac{4\alpha\gamma\lambda}{1-\rho} \hat{h} \right] T, \text{ and} \\
\kappa_4 &= \frac{12\alpha}{1-\rho} \left[\tilde{h} - \left(2 - \frac{\gamma}{\lambda} \right) \frac{4\alpha\gamma\lambda}{1-\rho} \hat{h} \right] T.
\end{aligned}$$

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