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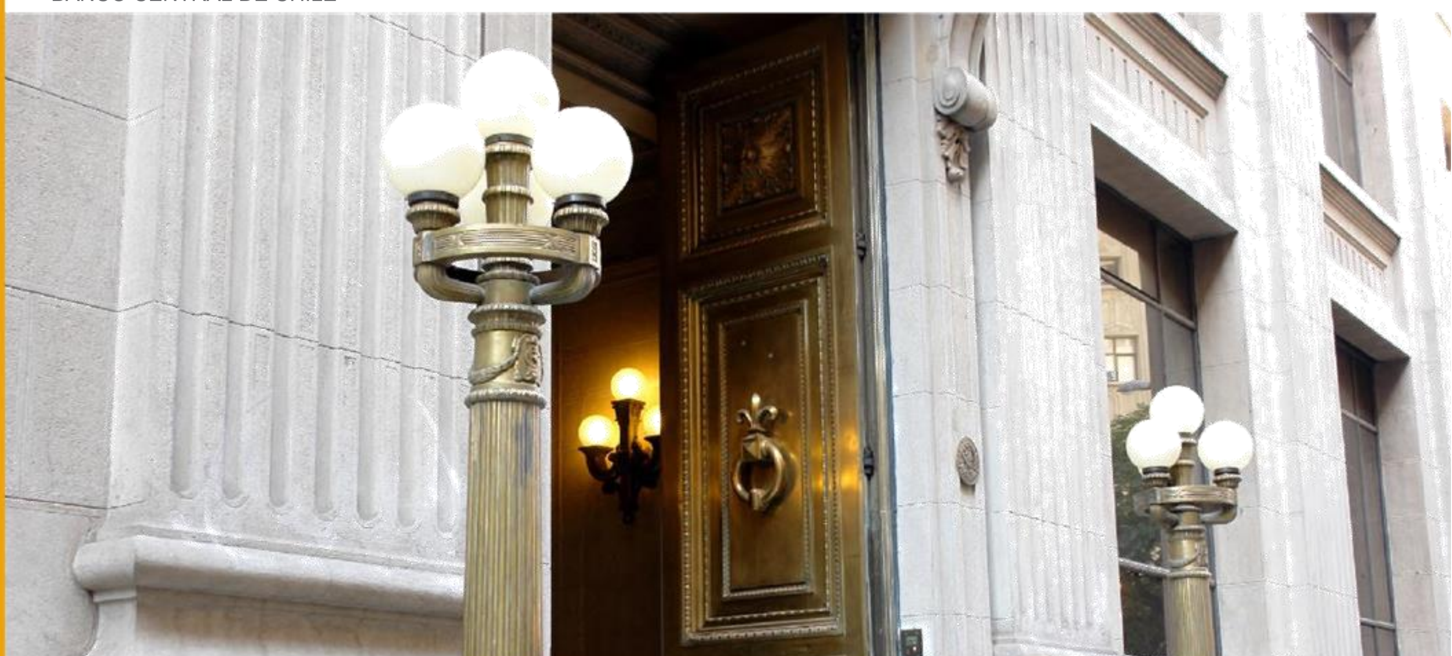

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Aggregating Distortions in Networks with Multi-Product Firms*

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Resumen

Investigamos el rol de las empresas multi-producto en la eficiencia de la asignación de recursos a través de cadenas productivas y su impacto en el crecimiento de la productividad total de los factores (PTF) agregada. Utilizando datos administrativos de transacciones de productos entre todas las empresas formales chilenas, proporcionamos evidencia de que shocks de demanda a un producto afectan la producción de otros productos dentro de la misma empresa, lo que sugiere que las empresas hacen producción conjunta, es decir, las empresas producen distintos productos ocupando insumos comunes. Desarrollamos un marco teórico para medir la eficiencia de la asignación de recursos en cadenas productivas con producción conjunta, derivando estadísticos suficientes no paramétricas para cuantificar estos efectos. Al aplicar el marco teórico con datos de Chile, encontramos que los efectos de reasignación de factores productivos, considerando la producción conjunta, explican el 86% del crecimiento observado de la PTF agregada para el período 2016-2022. Además, encontramos que ignorar la producción conjunta lleva a sobrestimar la importancia de reasignación de factores productivos en el crecimiento de la PTF agregada.

Abstract

We investigate the role of multi-product firms in shaping resource misallocation within production networks and its impact on aggregate total factor productivity (TFP) growth. Using administrative data on product transactions between all the for-mal Chilean firms, we provide evidence that demand shocks to one product affect the production of other products within the same firm, suggesting firms engage in joint production. We develop a framework to measure resource misallocation in pro-duction networks with joint production, deriving non-parametric sufficient statistics to quantify these effects. Applying the framework to Chile, we find that reallocation effects, considering joint production, explain 86% of observed aggregate TFP growth for the 2016-2022 period. Ignoring joint production leads to overestimating resource misallocation.

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1 Introduction

Resource misallocation across firms in production networks is considered a quantitatively relevant force driving aggregate Total Factor Productivity (TFP) growth. However, a common assumption is that firms produce and sell a single product. Using firm-to-firm product transaction data for the universe of formal firms in Chile, we find that 75% of firms sell multiple products, accounting for 99% of total firm-to-firm transaction value.

We test whether multi-product firms are a collection of independent single-product firms- if they engage in non-joint production-, as assumed often in the literature (Bernard et al. (2010, 2011); Hottman et al. (2016); Mayer et al. (2021)). We exploit the heterogeneous exposure of different product-buyer pairs within the same firm to COVID-19 lockdowns in Chile during the first quarter of 2020. We isolate firms that received direct demand shocks but did not suffer from any direct supply shock. We find that demand shocks to one product affect the production of other products within the same firm. This suggests that firms engage in joint production and aligns with recent findings by Boehm et al. (2022); Boehm and Oberfield (2023); Ding (2023)) that also challenge the non-joint production assumption.

Therefore, we build a theory to account for multi-product firms engaging in joint production within production networks and assess its aggregate TFP growth implications. We focus on understanding how multi-product firms shape resource allocations (allocative efficiency) and, hence, aggregate TFP growth. We bring the theory to the data using a granular database containing product-level information on prices and quantities traded between the universe of formal firms for Chile.

To characterize the joint production drivers of allocative efficiency, we derive nonparametric sufficient statistics. These statistics are constructed using observed data to decompose measured TFP into allocative efficiency and technology. Our theory assumes firms engage in joint production, generalizing Baqaee and Farhi (2020)'s production network structure.

We show that a buyer's position in the production network for a specific product, combined with that product's price change, determines how multi-product firms affect aggregate TFP growth. We measure the former using a "network distortion" statistic, which summarizes the distortions accumulated throughout a product's downstream supply chain. A price decrease in an upstream product absorbs distortions for the entire downstream supply chain that uses that product as an input, thereby improving allocative efficiency and aggregate TFP growth. We show that the firm-level effect can be calculated as the covariance between product-specific price changes and product-level network distortions.

To implement our framework, we use data on the universe of formal firms operating in Chile, sourced from the Chilean Internal Revenue Service. Due to tax enforcement requirements, all formal Chilean firms must declare their invoices with other firms, providing comprehensive information on all products, quantities, and prices traded between firms. We also access tax accounting declarations, offering data on each firm's revenue and input expenditures, including capital and labor.

While our framework allows any type of wedges, we assume that product-level output wedges (difference between price and marginal cost) are the sole source of inefficiency in the economy. To construct the network distortion, we need to measure two objects. The first is the interaction between a firm-product pairs network centrality measure and GDP share. We compute both components without needing any parametric assumption. The second is product-level markup levels, which we address using two different strategies. First, we estimate markups using an off-the-shelf production function approach (Dhyne et al. (2022)), where markups are affected by parametric assumptions. Second, to eliminate all parametric assumptions, we assume product markups equal firm-level average markup and quantify them using the accounting approach (revenue over cost). Quantitatively, both approaches generate nearly equivalent aggregate results.

We decompose measured aggregate TFP growth from 2016 to 2022 into reallocation effects and technology for Chile. Reallocation effects considering joint production explain 86% of the observed aggregate TFP growth. Ignoring joint production leads to overestimating resource misallocation. This can be interpreted as multi-product firms facing constraints in adjusting their product portfolios, limiting the scope for reallocation within the network.

We find that the contribution of joint production forces increases during economic disruptions. This suggests that product-level demand composition changed in response to economic disruptions, requiring firms to adjust their product portfolios. After the COVID-19 pandemic and subsequent high inflation, multi-product firms engaging in joint production increased their contribution to aggregate TFP growth. Ignoring technical constraints from joint production and implicitly assuming firms are single-product leads to overestimating reallocation forces. These results suggest that multi-product firms become particularly important in understanding aggregate TFP dynamics during periods of economic downturn.

Related Literature

This work contributes to and connects different strands of literature. We extend the rapidly developing literature on misallocation in production networks and growth accounting (e.g., Restuccia and Rogerson (2008); Hsieh and Klenow (2009); Baqaee and Farhi (2020); Bigio and La'O (2020); Osotimehin and Popov (2023)) by incorporating multi-product firms and joint production. Our theory provides a tool for growth accounting (Solow (1957); Hulten (1978); Basu and Fernald (2002); Petrin and Levinsohn (2012); Baqaee and Farhi (2020); Baqaee et al. (2023)) that decomposes aggregate TFP growth into technology, allocative efficiency under joint production in networks, generalizing existing methods to account for multi-product firms.

We contribute to the literature on multi-product firms (Bernard et al. (2010, 2011); Mayer et al. (2014); Hottman et al. (2016); Mayer et al. (2021)) by showing that these firms differ from collections of single-product firms in how they transmit shocks, due to their joint production activities. Our empirical strategy contrasts with other network studies (Boehm et al. (2019); Carvalho et al. (2020); Fujiy et al. (2022); Bai et al. (2024)) by focusing on how demand shocks to one product affect the production of other products within the same firm rather than examining downstream propagation of supply shocks.

We extend recent work on joint production (Boehm and Oberfield (2023); Carrillo et al. (2023); Ding (2023)) by revealing the allocative efficiency implications of joint production patterns in networks. [1] While these studies focus on the mechanisms by which joint production patterns are systematically linked to input proximity, our analysis takes these patterns as given and examines their implications for resource allocation and aggregate TFP growth.

In our empirical application, we use a comprehensive product-level trade database from Chile to quantify misallocation. This contrasts with much of the prior literature on production networks and misallocation, which uses industry-level input-output tables instead of firm-to-firm data. For example, Baqaee and Farhi (2020) impute U.S. Compustat data using an industry-level input-output table. Even when firm-level transaction data are available, the lack of complete price information often limits the analysis. For example, Kikkawa (2022) examines firm pair-specific markups based on a theoretical model using Belgian inter-firm transaction data. A contemporaneous study, Burstein et al. (2024),

[1] The literature on joint production contains seminal works by Powell and Gruen (1968); Diewert (1971); Lau (1972); Hall (1973, 1988). Despite its long-standing nature, this literature has recently been revived due to the recent availability of detailed firm and product-level data

uses the same dataset as ours but complements our work by analyzing misallocation arising from different prices of the same product to different buyers.

Lastly, this work is related to the literature on production function estimation. In particular, estimation methods for joint production have been developed recently Dhyne et al. (2017, 2022); De Loecker et al. (2016); Valmari (2023); Cairncross and Morrow (2023). While we have not developed any theoretical innovation in this area, our application of these methods is more comprehensive than that of previous studies. Unlike previous papers that estimate production functions for a specific industry or subsample of the economy, we apply the method of Dhyne et al. (2022) to estimate multi-product production functions, our estimation covers the universe of products traded in Chile by formal firms for the 2016-2022 period.

The rest of the paper is organized as follows. Section 2 presents the data and motivating facts, highlighting the dominance of multi-product firms and providing empirical evidence suggesting firms engage in joint production. Section 3 outlines the theoretical framework, deriving the non-parametric sufficient statistics for measuring allocative efficiency explained by multiproduct firms. Section 4 details the data and the construction of sufficient statistics. Section 5 applies the framework to decompose aggregate TFP growth in Chile for the 2016-2022 period, and Section 6 concludes.

2 Reduced form evidence

This section presents three empirical facts that motivate our theory of multi-product firms in production networks. We use data from the Chilean Internal Revenue Service (Servicio de Impuestos Internos, SII), covering all formal firms in Chile. In the cross-sectional analysis, we use the year 2018, a year not affected by any major shock. To test for joint production, we employ monthly data from January to April 2020, exploiting the unexpected nature of early COVID-19 lockdowns as a source of exogenous variation in product-specific demands.

The SII provides detailed information on firm-to-firm transactions through electronic tax documents. This dataset captures every product, quantity, and price traded between formal Chilean firms, containing data on over 15 million unique firm-specific product descriptions. [2] For 2018 we sum real-time quantity traded and value for every buyer

[2]The specific invoice variable is called "detail", which is inherently firm-specific and can differ between firms even for the same product. For example, one supermarket might declare selling "Sprite can 330cc"

firm-seller firm-detail triplet transaction. We divide the value traded over its quantity to obtain average yearly prices for every triplet. We use the 2018 data to briefly describe the main features of the firm-to-firm trade patterns in Chile.

Fact 1: Multi-product firms dominate domestic intermediate inputs trade

75% of firms produce multiple products, and these firms account for 98.94% of intermediate input transaction value. Table 1 illustrates the distribution of products per firm, weighted by firm-to-firm transaction values.

Table 1: Distribution of Product Numbers

Percentile	**Number of products (Unweighted)**	**Number of products (Weighted by transaction value)**
1%	1	1
5%	1	2
10%	1	4
25%	2	36
50%	7	475
75%	26	2,459
90%	119	32,195
95%	290	37,422
99%	1,253	62,372

Notes: The Table presents the distribution of product numbers for 2018. The left column shows the number of products without weighting, while the right column displays the number of products weighted by intermediate product transaction volumes of the firms.

while another declares selling "Sprite 330". This variation across sellers does not affect our analysis in this Section as we do not compare identical products across firms.

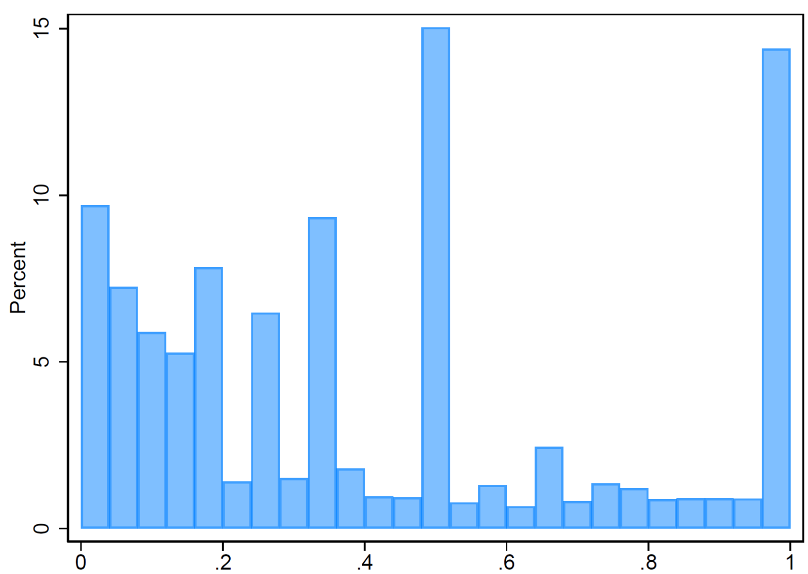
Fact 2. Multi-product firms sell different products to distinct sets of buyers

To characterize the heterogeneity from the intermediate inputs buyer perspective across products and within firms, we construct the following measure:

$$S_i = \frac{\text{number of buyers of the main product of firm } i}{\text{number of buyers of firm } i},$$

where the main product is defined as the product with the largest sales within firm i in 2018. Figure 1 presents the distribution of this measure across firms. If the buyers of the seller's main product and its other products were exactly the same, S_i would be one. Nevertheless, while there is some mass at $S_i = 1$, for more than 50% of multi-product firms, buyers of their main product constitute less than 50% of their total buyer firms base.

Figure 1: Buyer heterogeneity across product within firm



Notes: Histogram of the number of buyers buying the main product of firm i / number of buyers in firm i, for a multi-product firm. Main product is defined as the product with the highest sales within that firm. Data are from 2018.

This heterogeneity implies that each product have different buyers downstream and hence builds different sub-production networks, potentially subject to different distortions.

Fact 3: Demand shocks to one product affect the production of other products within the same firm

Given the prevalence of multi-product firms, we examine their role in production networks. Many existing studies (Bernard et al. (2010, 2011); Hottman et al. (2016); Mayer et al. (2021)) treat multi-product firms as collections of independent single-product firms. Under this assumption, multi-product firms can be relabeled as multiple fictitious firms and analyzed using canonical production network models. We test the validity of this non-joint production assumption and determine whether multi-product firms differ from collections of single-product firms.

We adapt the statistical test for non-joint production proposed by Ding (2023) to the context of production networks. We exploit heterogeneous exposure to local buyer shocks for each firm's different products. While many studies focus on supply shocks propagating downstream, we examine how demand shocks to specific products affect the production of other products within the same firm. We use monthly data for the first four months of 2020 to capture the initial impact of the COVID-19 pandemic. We treat different establishments of the same firm as different firms to treat regional differences in Covid lockdown as exogenous shocks. Monthly, we sum quantity traded and value for every buyer firm-seller firm-detail triplet transaction. We divide the value traded over its quantity to obtain average monthly prices for every triplet.

COVID-19 Lockdowns

Between March and April 2020, the Chilean government declared lockdowns for a subset of the 346 Chilean counties. Like in many other countries, lockdowns were unexpected and initially thought to be temporary. While lockdown policies later spread to 294 out of 346 counties in Chile, we focus on this initial period in March and April to ensure the shock was unanticipated. Figure 2 shows regions that experienced lockdowns in April 2020, illustrating the sparse initial distribution of lockdowns across the country.

Figure 2: Distribution of Early Covid-19 lockdown in Chile

Notes: Lockdown counties as of April 2020 are red; all others are gray.

We hypothesize that lockdowns reduced intermediate input transactions from firms in unshocked (gray) regions to buyers in red regions (early COVID lockdowns). We consider this an unanticipated intermediate goods demand shock to specific products of firms in non-closed regions. To study reduced intermediate goods purchases from affected counties, we first estimate the following reduced-form specification at the buyer level:

$$\log \text{Intermediate Input Purchases}_{it} = \alpha Lockdown_{it} + FE_t + FE_i + \varepsilon_{it}, \tag{1}$$

As a threat to identification, there may be a bias in the coefficient if firms in lockdown areas are more likely to purchase from suppliers also located in lockdown areas. To address this concern, we also report results from a restricted sample that includes only firms whose suppliers are all located in non-lockdown areas.

Table 2: Lockdown and intermediate input purchases

	(1)	(2)	(3)
Lockdown Dummy	-0.222***	-0.230***	-0.191***
	(0.0524)	(0.00521)	(0.0589)
Firm FE	Y	Y	Y
Time FE	N	N	Y
Sector × Time FE	N	Y	N
Restricted sample	N	N	Y
Observations	4,345,534	4,345,534	378,646

Notes: The Table reports the results of estimating equation (1) by OLS, clustered at the firm-municipality level. The sample periods are January to April 2020. Columns (1) and (2) report results for the full sample, while column (3) reports results restricted to firms where none of the suppliers are located in the lockdown area. Three stars indicate statistical significance at the 1% level.

The coefficient of interest is negative, indicating purchases of intermediate inputs from lockdown counties decreased by about 20% on average. We cannot reject the null hypothesis of the existence of a negative demand shock to intermediate inputs sold by firms located in lockdown regions.

Non-joint production test

To test for non-joint production, we test the following null hypothesis: a given product sales are only affected by direct demand shocks; they are unaffected by other product demand shocks within the same firm. We define a firm's main product as the product with the highest sales in January- February 2020. We construct a treatment indicator $lockdown_{igt}$ for each product g of seller firm i in month t:

$$lockdown_{igt} = \begin{cases} 1 & \text{if at least one buyer for product } g \text{ is in a lockdown area} \\ 0 & \text{otherwise} \end{cases}$$

We denote this as $lockdown_{imt}$ for the main product m. For all products $g \neq m$, we estimate:

$$\log sales_{igt} = \alpha Lockdown_{imt} + \beta Lockdown_{igt} + \boldsymbol{\gamma}' \boldsymbol{X}_{it} + FE_t + FE_{ig} + \varepsilon_{igt}, \tag{2}$$

The coefficient of interest is α. We focus on whether α significantly differs from zero, allowing us to reject non-joint production. β is a control term and should be negative regardless of joint production. To estimate α controlling for potential bias, $Lockdown_{imt}$ must be conditionally orthogonal to the error term. The error term may contain supply-side shocks correlated with $lockdown_{imt}$ if suppliers and main product buyers are likely to be in the same location. We leverage the dataset granularity and impose sample restrictions to isolate demand-side shocks by excluding supply-side lockdown effects:

1. The seller firm's county is located in a non-lockdown area.
2. None of the seller firm's suppliers (defined in January and February before lockdowns took place) are in lockdown areas.

Additionally, we include firm-specific input price indices as a control variable to account for other potential supply shocks unrelated to lockdowns. [3] Table 3 presents the main regression results. First, β is negative as expected and close to the magnitude anticipated from Table 2 (about 20%), suggesting that the constructed shocks are empirically working as negative demand shifters. The coefficient α is statistically significant and positive across all specifications, allowing us to reject the null hypothesis of non-joint production. This implies that when a firm's non-main products are exposed to lockdown shocks, sales of those non-main and non-shock-affected products increase.

[3] The Tornquvist price index was constructed using input prices using January 2020 is the base month.

Table 3: Non-Joint Production Test

	(i)	(ii)	(iii)	(iv)
Main product's buyer lockdown	0.093***	0.089***	0.092***	0.073**
	(0.027)	(0.028)	(0.028)	(0.028)
Own buyer lockdown	-0.194***	-0.205***	-0.202***	-0.236***
	(0.035)	(0.036)	(0.036)	(0.039)
Control	N	N	Y	Y
Firm × product FE	Y	Y	Y	Y
Time FE	Y	N	N	N
Product class × Time FE	N	N	Y	Y
Observations	298,252	286,918	289,457	278,276

Notes: The Table reports the results of estimating equation (2) by OLS, clustering standard errors at the firm-county level. The sample periods are January to April 2020. Two and three stars indicate statistical significance at the 95% and 99% levels, respectively.

Discussion on other within-firm spillover mechanisms

Our empirical results reject the hypothesis of non-joint production in multi-product firms, suggesting that demand shocks to one product affect the production of other products within the same firm. We compare our findings with three relevant studies and highlight the implications of our results.

Almunia et al. (2021) propose a decreasing return to scale (DRS) or firm-specific factor model to explain how domestic demand declines affect export behavior in Spain. Their model predicts that a negative demand shock in one market for a given product will have a positive effect on sales of the same product in other markets. However, their model remains silent on the effects across different products within a firm.

Ding (2023) examines joint production effects in the US using 5-year Census data, focusing on industries sharing knowledge-intensive inputs. While Ding shows positive spillovers across industries sharing intangible inputs, this mechanism is unlikely to explain our results. The differences in time horizon (5 years vs. monthly data) and R&D intensity (Chile's R&D spending is less than one-tenth of the US's as a percentage of GDP) limit its applicability to our context. Moreover, in our formulation, such complementarities would predict $\alpha < 0$, contrary to our finding of $\alpha > 0$.

Giroud and Mueller (2019) models demand-driven regional spillovers based on financial constraints using US multi-region firm data. In this model, firms facing credit constraints optimize resource allocation across regions. A negative demand shock in one region leads to employment reductions in other regions due to shared financial constraints within the firm. This mechanism predicts a negative α, as a negative shock to one product would lead to reduced production of other products through the common financial constraint. However, our results show positive spillovers ($\alpha > 0$) across products, inconsistent with this financial constraint mechanism in the context of COVID-19 demand shocks.

In sum, our results suggest that multi-product firms are not collections of independent product lines, which has non-trivial implications for the mechanisms by which demand shocks propagate throughout production networks. These empirical results motivate the theoretical framework we develop in the following section.

3 A theory to aggregate distortions in networks with multi-product firms

We propose a theoretical framework to analyze resource misallocation in multi-product firms' production networks. This framework rationalizes our empirical findings on the prevalence and characteristics of multi-product firms in production networks, generalizing the work of Hulten (1978) and Baqaee and Farhi (2020). We characterize the first-order effects of firm-product-level shocks on aggregate total factor productivity (TFP) growth in an economy with arbitrary wedges, product-level production networks, and joint production.

3.1 Joint production

Our empirical results show that demand shocks to specific products within firms affect the production of other non-shocked products within the same firm. To capture this, we adopt a joint production setup, where firms use common inputs to produce different products, allowing some inputs to be used in multiple products simultaneously.

Following Hall (1973), let $J(\boldsymbol{q}, \boldsymbol{x})$ be a joint production function, where $\boldsymbol{q}$ is a vector of outputs and $\boldsymbol{x}$ is a vector of inputs. The joint cost function, derived from the firm's cost minimization problem, is:

$$C(\boldsymbol{q}, \boldsymbol{p}) \equiv \min_{x \in V(q)} \boldsymbol{p}'\boldsymbol{x},$$

where $V(\boldsymbol{q})$ is the input requirement set, $V(\boldsymbol{q}) = \{\boldsymbol{x} | J(\boldsymbol{q}, \boldsymbol{x}) \geq 0\}$ and $\boldsymbol{p}$ is a vector of input prices. We introduce two assumptions about the shape of a joint production function which will be used throughout this paper.

Assumption 1. Constant return to scale (CRS): $J(\boldsymbol{q}, \boldsymbol{x}) = 0$ implies $J(\lambda\boldsymbol{q}, \lambda\boldsymbol{x}) = 0$ for any scalar λ.

Unlike a single-output production function, the output is a vector. Note that this does not assume CRS for each single-output production function.

Assumption 2. Separability between input and output bundles: The joint production function can be written as $J(\boldsymbol{q}, \boldsymbol{x}) = -F^Q(\boldsymbol{q}) + F^X(\boldsymbol{x})$, and the joint cost function as $C(\boldsymbol{q}, \boldsymbol{x}) = H(\boldsymbol{q})\varphi(\boldsymbol{p})$.

Note that this is different from assuming non-joint production functions when a firm is multi-product. In that case, the output q is not a vector but a single product and thus degenerates to $F^Q(q) = q$. In example 1, we illustrate a joint production function satisfying Assumptions 1 and 2:

Example 1. Constant-Elasticity of Transformation output bundle and Constant-Elasticity of Substitution Input bundle (CET-CES):

$$\underbrace{\left(\sum_g q_g^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}}}_{\text{Output bundle}} = A \underbrace{\left(L^{\frac{\theta-1}{\theta}} + K^{\frac{\theta-1}{\theta}}\right)^{\frac{\theta}{\theta-1}}}_{\text{Input Bundle}}$$

The associated cost function is:

$$C(\boldsymbol{q}, w, r) = \frac{1}{A}\left(\sum_g q_g^{\frac{\sigma+1}{\sigma}}\right)^{\frac{\sigma}{\sigma+1}} \left(w^{1-\theta} + r^{1-\theta}\right)^{\frac{1}{1-\theta}}$$

where L and K are the two inputs, and w and r are their prices and $\boldsymbol{q}$ is a vector of outputs.

The input bundle takes a standard CES function, but the output bundle is a vector of products rather than a scalar. This functional form is discussed in Hall (1973), where the parameter σ is called the constant elasticity of transformation. It gives a constant value to the production possibility frontier's curvature of the products within a firm. This example is only illustrative as our theoretical framework does not require any parametric assumption.

3.2 Network Setup

We use Baqaee and Farhi (2020) 's input-output notation and definitions to present our generalization and add product-level (instead of firm-level) objects. In the absence of joint production, every product can be considered a fictitious firm so that Baqaee and Farhi (2020) setup applies.

Multi-Product Firms

Firm $i \in \mathcal{N}$ produces product $g \in \mathcal{G}$ and uses products $g' \in \mathcal{G}$ from other firms $j \in \mathcal{N}$ and factors (Labor, L and Capital, K) as production inputs. [4] We assume a production set with CRS and separability:

$$F_i^Q\left(\underbrace{\left\{q_{ig}\right\}_{i\in N, g\in G}}_{\text{outputs}}\right) = A_i F_i^X\left(\underbrace{\left\{x_{i,jg'}\right\}_{j\in N, p\in G}}_{\text{Intermediate product } g' \text{ from } j}, L_i, K_i\right),$$

Firms charge a product-specific markup μ_{ig} over its product-specific marginal cost, so that the prices is defined as $p_{ig} = mc_{ig}\mu_{ig}$.

Final Demand

There is a representative household with homothetic utility function $U\left(c_{ig}, ..., c_{NG}\right)$ that receives income from factor payments and profits from firms they own, following a budget constraint:

$$\sum_{i\in\mathcal{N}}\sum_{g\in\mathcal{G}} p_{ig}c_{ig} = \sum_{f\in\{L,K\}} w_f L_f + \sum_{i\in\mathcal{N}}\sum_{g\in\mathcal{G}}\left(1 - 1/\mu_{ig}\right)p_{ig}q_{ig}$$

[4] We treat factors exhibiting zero return to scale production functions; they generate production inputs without using inputs from other firms.

Each product can either be consumed by final consumers (c_{ig}) or used as an input in production by other firms ($x_{ji,g}$), facing the following resource constraint:

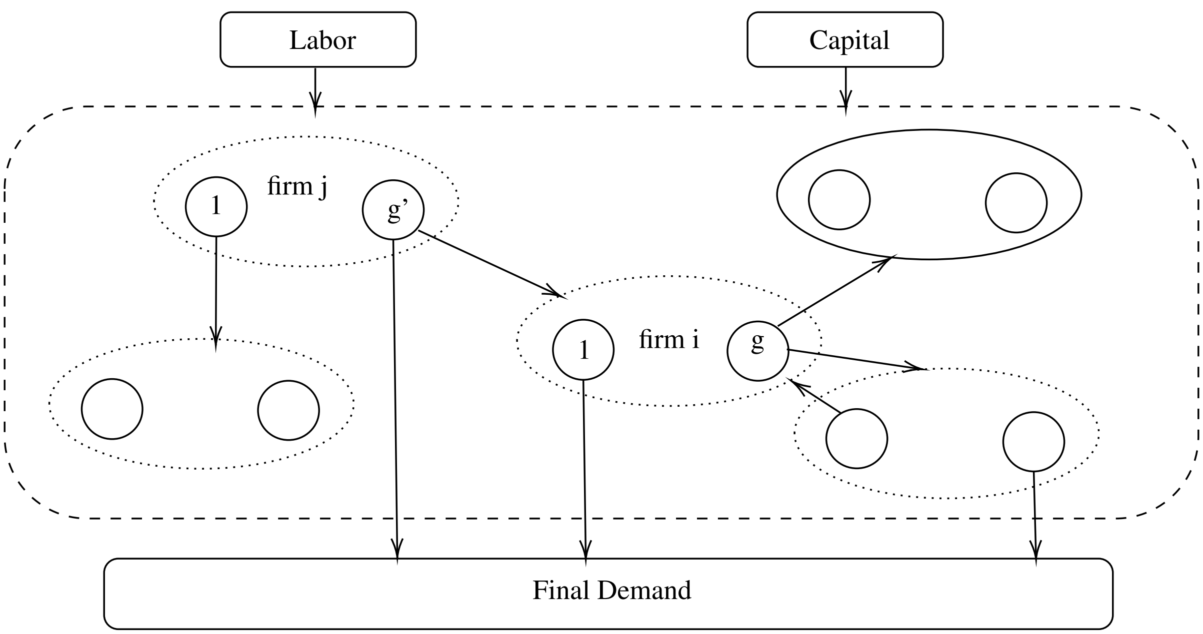
$$q_{ig} = c_{ig} + \sum_{j \in \mathcal{N}} x_{jig}, \quad \sum_{i \in \mathcal{N}} L_i = L, \quad \sum_{i \in \mathcal{N}} K_i = K$$

A stylized representation is given in Figure 3 showing the flow of products.

General Equilibrium

Given a vector of firm-level productivity $\boldsymbol{A}$ and vector of product-level markups $\boldsymbol{\mu}$ for all $i \in \mathcal{N}$ and $g \in \mathcal{G}$, the general equilibrium is a set of prices p_{ig}, intermediate input choices $x_{ijg'}$, factor input choices L_i, K_i, output q_{ig}, and consumption choices c_{ig}, such that: (i) the price of each product is equal to its markup multiplied by its marginal cost; (ii) households maximize utility under budget constraints, given prices; and (iii) markets clear for all products and factors.

Figure 3: Graphical illustration of networks with multi-product firms



Notes: The dashed line represents firms' universe $\mathcal{N}$, the dotted circled line represents each firm's boundary, and the circled line represents each product within a firm. The two top nodes represent factors, and the bottom node represents households. Arrows represent the direction of input flows.

3.3 Input-Output Definitions

To state our decomposition results, we introduce notation for input-output objects.

Product-Level Input-Output Matrix

The product-level input-output matrix $\tilde{\Omega}$ is a $(\mathcal{N}\mathcal{G} + \mathcal{F})$ square matrix where $\mathcal{N}$ is the number of firms, $\mathcal{G}$ is the number of products, and $\mathcal{F}$ is the number of factors. $\tilde{\Omega}$ has at its ig, jg'^{th} element the expenditure share of product g' from firm j and factor $f \in \mathcal{F}$ used by firm i in production over firm i total costs (of producing all its products). From the separability assumption, the same expenditure share applies for all products, g that firm i produces. Thus, $\tilde{\Omega}_{ig,jg'}$ and $\tilde{\Omega}_{ig,f}$ are as follows.

$$\tilde{\Omega}_{ig,jg'} = \frac{p_{jg'}x_{i,jg'}}{\sum_{j,p} p_{jg'}x_{i,jg'} + \sum_f w_f L_{if}}, \quad \tilde{\Omega}_{ig,f} = \frac{w_f L_{if}}{\sum_{j,p} p_{jg'}x_{i,jg'} + \sum_f w_f L_{if}}$$

The product cost-based Leontief inverse $\tilde{\Psi}$ captures the direct and indirect cost exposures of each firm-product pair through production networks. Each element of $\tilde{\Psi}$ measures the weighted sum of all paths between any two non-zero firm-product pairs.

$$\tilde{\Psi} \equiv (I - \tilde{\Omega})^{-1} = I + \tilde{\Omega} + \tilde{\Omega}^2 + \dots$$

We define the value-added share vector b to be:

$$b_{ig} = \begin{cases} \frac{p_{ig}c_{ig}}{GDP} & \text{if } i \in \mathcal{N}, g \in \mathcal{G} \\ 0 & \text{otherwise} \end{cases}$$

We set GDP to be the numeraire and we define the product-level cost-based Domar weight, $\tilde{\lambda}_{ig}$.[5] This measures the importance of product g from firm i in final demand in two dimensions, directly when it is sold to final consumers and indirectly through the production network when it is sold to other firms that, eventually downstream production networks will reach final consumers.

$$\tilde{\boldsymbol{\lambda}}' \equiv \boldsymbol{b}'\tilde{\Psi} = \boldsymbol{b}' + \boldsymbol{b}'\tilde{\Omega} + \boldsymbol{b}'\tilde{\Omega}^2 + \dots$$

[5] We denote $\tilde{\Lambda}_f$ with $f \in \{L, K\}$.

Firm-Level Aggregation

Summing over products by firms, we recover the firm-level cost-based Domar weight $\tilde{\lambda}_i$, which we use to compute the within-firm product-level Domar weight share s_{ig}:

$$\tilde{\lambda}_i = \sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig}, \quad s_{ig} = \frac{\tilde{\lambda}_{ig}}{\tilde{\lambda}_i},$$

Finally, we define firm-level aggregate markup as:

$$\mu_i = \frac{\text{sales of } i}{\text{total cost of i}},$$

National Accounts

GDP is defined as the sum of all product values consumed by final consumers: $GDP = \sum_{i \in \mathcal{N}} \sum_{g \in \mathcal{G}} p_{ig} c_{ig}$. Real GDP (Y) changes can be computed as:

$$d \log Y = d \log GDP - \sum_{i \in \mathcal{N}} \sum_{g \in \mathcal{G}} \frac{p_{ig} c_{ig}}{GDP} d \log p_{ig},$$

Factor shares are defined as:

$$\Lambda_L = \frac{wL}{GDP}, \quad \Lambda_K = \frac{rK}{GDP}$$

3.4 Network Distortion

With all the needed ingredients, we now define the network distortion, Γ_{ig}, which is the key input to the sufficient statistic strategy we propose. The network distortion is defined as the ratio of the product cost-based Domar weight, $\tilde{\lambda}_{ig}$ to the product sales share (to GDP), adjusted by the product-level markup.

Definition 1. Network Distortion

$$\Gamma_{ig} \equiv \underbrace{\frac{\tilde{\lambda}_{ig}}{\textit{sales share}_{ig}}}_{\text{downstream distortion}} \times \underbrace{\mu_{ig}}_{\text{own markup}},$$

It summarizes the cumulative distortion in the downstream supply chain of product

g sold by firm i. A product cost-based Domar weight equals observed sales shares in efficient economies. In an inefficient economy, however, a portion of the indirect demand transmitted from downstream firm-product pairs to upstream firm-product firms is absorbed as profit by downstream firms. This effect accumulates in each supply chain transaction upstream and continues until indirect demand reaches product g sold by firm i. As a result, the sales share of a product is smaller relative to an efficient economic outcome.

Thus, the larger the ratio is, the greater the cumulative distortions in the downstream supply chain. In an efficient economy, there are no markups, and the product cost-based Domar weight equals the product sales share, and hence, $\Gamma_{ig} = 1$ for all i and g.

Next, we define the relative product network distortion, which ranks product distortions within a given firm. It is measured as the relative downstream distortion of product g with respect to the average distortion of all products within firm i, Γ_i.

Definition 2. Relative Network Distortion

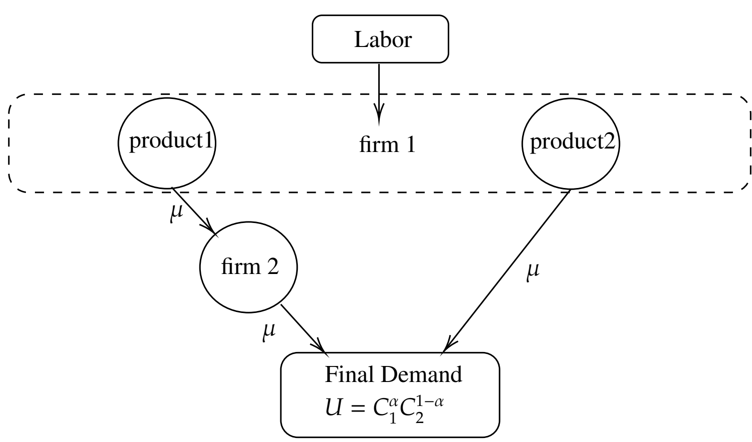
$$\gamma_{ig} \equiv \frac{\Gamma_{ig}}{\Gamma_i}$$

where the average distortion of firm i is defined as: $\Gamma_i \equiv \sum_g \tilde{\lambda}_{ig} / \sum_g \left(salesshare_{ig}/\mu_{ig} \right)$.

A Simple Example of Network Distortion

To illustrate the concept of network distortions and its computation, we provide an example of a simplified economy composed of two firms and a representative household.

Figure 4: A simplified economy with production networks and multi-product firms



The household has Cobb-Douglas preferences:

$$U = c_1^{\alpha} c_2^{1-\alpha},$$

with an inelastic labor supply. GDP is normalized to 1, so final consumption expenditure shares are determined by α. We specify the following production structure:

- Firm 1 uses labor to produce two differentiated products.
- Product 1 is sold to firm 2. Product 2 is sold directly to households. Both products have the same markup μ.
- Firm 2 uses product 1 from firm 1 using a linear technology and sells it to households with a markup μ.

In this setup, sales (shares) to final consumption are α for product 1 and $1-\alpha$ for product 2. However, firm 1's sales of product 1 are reduced by the markup charged by firm 2, which is $(1-\alpha)/\mu$.

Product cost-based Domar weights are α for product 1 of firm 1 and $1-\alpha$ for product 2 of firm 2. However, there is no need to discount the markup when calculating firm 1's derived demand for product 2. In matrix notation, the value-added share vector and the product cost-based input-output matrix are:

$$b = \begin{bmatrix} \alpha \\ 0 \\ 1-\alpha \\ 0 \end{bmatrix}, \quad \tilde{\Omega} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the components of the matrix and vector are arranged in the following order: product 1 and 2 of firm 1, firm 2, and labor.

The cost-based Domar weights are:

$$\begin{aligned} \tilde{\boldsymbol{\lambda}}' &= \boldsymbol{b}' + \boldsymbol{b}'\tilde{\Omega} + \boldsymbol{b}'\tilde{\Omega}^2 + \dots \\ &= \underbrace{\begin{bmatrix} \alpha, & 0, & 1-\alpha, & 0 \end{bmatrix}}_{\text{Final demand}} + \underbrace{\begin{bmatrix} 0, & \alpha, & 0, & 0 \end{bmatrix}}_{\text{Indirect demand}} \\ &= \begin{bmatrix} \alpha, & \alpha, & 1-\alpha, & 0 \end{bmatrix} \end{aligned}$$

These weights represent the counterfactual sales shares if markups were removed while keeping expenditure shares constant.

Following the definition, the network distortion is: $\Gamma_{ig} = \frac{\tilde{\lambda}_{ig}}{\text{sales share}_{ig}} \times \mu_{ig}$ for firm 1:

$$\Gamma_{11} = \frac{\alpha}{\alpha/\mu}\mu = \mu^2, \qquad \Gamma_{12} = \frac{1-\alpha}{(1-\alpha)}\mu = \mu,$$

The following Table 4 summarizes the results:

Table 4: Sales share, cost-based Domar weight and network distortion in this example

	product 1 of firm 1	product 2 of firm 1
(1) Sales share	α/μ	$1-\alpha$
(2) Cost-based Domar weights	α	$(1-\alpha)$
(3) Network Distortion: (2)/(1) $\times$ own markup	μ^2	μ

While the markup of product 2 from firm 1 and the product from firm 2 equal μ, product 1 from firm 1 has a larger network distortion of μ^2 than that of product 2. It reflects not only the product's own markup but also the downstream distortions faced by the product. In this case, product 1 from firm 1 generates a distortion by charging a markup and is subject to an additional distortion downstream production networks as firm 2 uses a marked-up input on its production. The sum of both distortions is the main driver when characterizing the multi-product channel of allocative efficiency, which is introduced in the next section.

3.5 Aggregation Theorem with multi-product firms within production networks

In this section, we generalize the concept of an allocation matrix introduced by Baqaee and Farhi (2020) to allow for joint production and derive sufficient statistics.

Let $\mathcal{X}$ be an $(\mathcal{N}+\mathcal{F})\times(\mathcal{N}\mathcal{G}+\mathcal{F})$ admissible input allocation matrix, where the columns are buyer firms and the rows are seller-product pairs. Each of its elements $\mathcal{X}_{ijg} = \frac{x_{ijg}}{q_{jg}}$ is the share of the output of product g produced by firm j that firm i uses as a production input.

A productivity shock ($\mathrm{d}\log A$) and a markup shock ($\mathrm{d}\log \mu$) effect in real GDP, Y can be decomposed into a pure change in productivity ($\mathrm{d}\log A$) for a given fixed allocation matrix $\mathcal{X}$ and the change in the distribution of $\mathcal{X}$ ($\mathrm{d}\mathcal{X}$) holding productivity constant. In vector notation:

$$\mathrm{d}\log\mathcal{Y} = \underbrace{\frac{\partial\log\mathcal{Y}}{\partial\log A}\mathrm{d}\log A}_{\Delta\text{ Technology}} + \underbrace{\frac{\partial\log\mathcal{Y}}{\partial\mathcal{X}}\mathrm{d}\log\mathcal{X}}_{\Delta\text{ Allocative Efficiency}} \tag{3}$$

We now present a decomposition of changes in aggregate TFP that accounts for multi-product firms and arbitrary production networks with product-level distortions.

Proposition 1. *Growth accounting in production networks with multi-product firms: To a first order, aggregate TFP can be decomposed into technology and allocative efficiency terms as follows:*

$$d\log TFP = \underbrace{\sum_{i\in\mathcal{N}}\tilde{\lambda}_i d\log A_i}_{\Delta\ Technology} + \underbrace{\underbrace{\sum_{i\in\mathcal{N}}\tilde{\lambda}_i Cov_{s_i}\left(d\log\boldsymbol{p}_i, \frac{1}{\boldsymbol{\gamma}_i}\right)}_{Multi\text{-}product\ term} - \underbrace{\sum_{i\in\mathcal{N}}\tilde{\lambda}_i d\log\mu_i}_{Firm\ level\ Markup} - \underbrace{\sum_{f\in\mathcal{F}}\tilde{\Lambda}_f d\log\Lambda_f}_{Aggrregate\ Factor\ Shares}}_{\Delta\ Allocative\ Efficiency} \tag{4}$$

where $\boldsymbol{\gamma}_i = (\gamma_{i1}, ..., \gamma_{iG})$ *and* $d\log\boldsymbol{p}_i = (d\log p_{i1}, ..., d\log p_{iG})$.

The proof can be found in Appendix E. The change in aggregate TFP can be decomposed into two parts: a technology term and an allocative efficiency term. The technology term represents a weighted average of firm-level Hicks-neutral productivity changes, using cost-based Domar weights. The allocative efficiency term is further decomposed into three components: a multi-product firm term, a change in aggregate factor shares, and firm-level average markup changes. We provide an interpretation of each term:

The multi-product term captures the allocative efficiency implications of firm-level product portfolio choice adjustments. The contribution to the allocative efficiency of each product depends on the change in the price of the product relative to the average price change in that firm and the relative network distortion of that product in the firm's product portfolio. Thus, for example, if the price change for a given product g in firm i is higher than the average price change in firm *i*, and product g is more distorted relative to the average product within firm *i*, the covariance will be positive. Intuitively, a decline in the price of an upstream product absorbs the distortion for all downstream firms that use that product as an input (directly or indirectly). The covariance implies that this effect is greater when prices of relatively more distorted products fall.

We now turn to the factor shares and firm-level markup terms. If the initial equilibrium is inefficient, the products charging markups are under-produced relative to an efficient economy. Improving the allocation involves a reallocation of resources to a more distorted part of the economy; firms-product pairs that charge relatively high markups. A decrease in factor shares implies a reallocation of resources to the portion of the economy with relatively high markups. However, if the change in factor share is due to a change in markup, this is a mechanical change and does not imply reallocation, so the contribution of the change must be purged; this is what the firm-level markup term captures. Factor shares and firm-level markup terms are the terms proposed by Baqaee and Farhi (2020); both terms are also valid under a joint production approach and, together with the multi-product term this work introduces constitutes allocative efficiency.

Relation to existing aggregation theorems:

We show Proposition 1 nest existing aggregation theorem for production networks as a special case.

Corollary 1. *Baqaee and Farhi (2020): If all firms do not engage in joint production and impose the same markup on all their products (single product firms assumption), then to a first-order, aggregate TFP growth can be decomposed into technology and allocative efficiency terms as follows.*

$$d \log TFP = \underbrace{\sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log A_i}_{\textit{Technology}} \underbrace{- \sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log \mu_i - \sum_{f \in \mathcal{F}} \tilde{\Lambda}_f d \log \Lambda_f}_{\textit{Allocative Efficeincy}}$$

The proof follows from the fact that the covariance term from proposition 1 is zero because the change in marginal cost and markup for all products within a firm are equal.

Our approach quantifies misallocation through the multi-product channel by measuring deviations from the single-product, single-markup assumption when product-level data is available. If this assumption holds, the multi-product term becomes zero. While the assumption of uniform marginal costs and markups is unlikely to hold in practice, its quantitative relevance remains an empirical question. Our decomposition not only quantifies the extent to which this assumption is violated but also isolates the impact of existing misallocation literature.

Finally, in the absence of markups, when prices equal marginal costs, allocative efficiency converges to zero. In this case, all aggregate TFP changes are attributed to technology, aligning with Hulten (1978).

Corollary 2. *Hulten (1978): Growth accounting in an efficient economy:*

$$d \log TFP = \underbrace{\sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log A_i}_{\text{Technology}}$$

The proof follows from the fact that the markup is always 1, the markup change term is zero, and the sum of factor shares is always 1, hence the sum of factor changes is always zero, and the covariance of the multi-product term is zero because $\boldsymbol{\gamma}_i = (\gamma_{i1}, ..., \gamma_{iG})$ are all 1 in an efficient economy.

When the economy is efficient, Proposition 1's formula converges to Hulten's theorem. That is, measured aggregate TFP growth equals the Domar weighted sum of firm-level productivity changes.

Illustrative Examples We revisit the example from the network distortion subsection to build intuition about Proposition 1. In this economy, Firm 1 employs workers to produce two differentiated products. Firm 1 sells product 1 to firm 2 with a markup μ, while firm 2 processes this product using linear technology and sells it to households with an additional markup μ. Concurrently, firm 1 sells product 2 directly to households, also applying a markup μ. For simplicity, we set $\alpha = 1/2$, and hence the representative household utility function is $U = c_1^{1/2} c_2^{1/2}$

To illustrate how Proposition 1 operates, let's consider two examples. Consider a negative markup shock $-\epsilon$ for product 1 of firm 1. In the first example, firms produce according to a non-joint production structure, while in the second example, firms engage in joint production, which is parameterized with a constant elasticity of transformation (CET) technology. Appendix E provides the proofs for both examples.

Example 1: Non-joint Production In this example, we model firm 1's production technology as non-joint, meaning each product is produced independently using only labor. The production functions, with productivity normalized to 1 are:

$$q_{11} = L_{11}, \quad q_{12} = L_{12},$$

Firm 2 uses linear technology to process and sell product 1 to the household:

$$q_{21} = q_{11}$$

We now apply Proposition 1. In this example, the marginal cost change is the same for products 1 and 2, so the relative price is determined by the difference in markup changes.

First, we compute the multi-product term:

$$\begin{aligned} Cov_{s_i}\left(d\log \boldsymbol{p}_i, \frac{1}{\gamma_i}\right) &= Cov_{\left[\frac{1}{2},\frac{1}{2}\right]}\left(\begin{bmatrix} -\epsilon \\ 0 \end{bmatrix}, \begin{bmatrix} 2\frac{1}{(\mu+1)} \\ 2\frac{\mu}{(\mu+1)} \end{bmatrix}\right) \\ &= \left(\frac{\mu-1}{2(1+\mu)}\right)\epsilon > 0 \end{aligned}$$

This result indicates that the decrease in the price of product 1, which has a larger network distortion, has removed downstream distortions and improved allocative efficiency.

The remaining terms are:

$$-\underbrace{\sum_{i\in\mathcal{N}} \tilde{\lambda}_i d\log \mu_i}_{\text{Firm level Markup}} - \underbrace{d\log \Lambda_f}_{\text{Aggregate Labor Shares}} = -\frac{1}{\mu+1}(-\epsilon) - \left(-\frac{1}{\mu+1}\right)(-\epsilon) = 0$$

Therefore,

$$\Delta \log TFP = \left(\frac{\mu-1}{2(1+\mu)}\right)\epsilon > 0$$

Overall, these results generate a positive change in allocative efficiency. [6]

Example 2: Joint Production. In this example, we employ the joint production function introduced in the network distortion subsection example.

$$\left(q_{11}^{\frac{\sigma-1}{\sigma}} + q_{12}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = L_1$$

The relative price change is given by:

$$d\log p_1/p_2 = \left(\frac{\sigma}{\sigma+1}\right) d\log \mu_1/\mu_2 + \frac{1}{\sigma+1} d\log \lambda_1/\lambda_2$$

[6] As shown in Appendix B, in this non-joint production setting, Baqaee and Farhi (2020)'s formula generates the same result.

Applying our decomposition, we find:

$$
\begin{aligned}
d \log TFP &= Cov_{s_i}\left(d \log \boldsymbol{p}_i, \frac{1}{\gamma_i}\right) \\
&= \left(\frac{\sigma}{\sigma+1}\right)\left(\frac{\mu-1}{2(1+\mu)}\right)\epsilon > 0
\end{aligned}
$$

It converges to example 1 in the limit ($\sigma \to \infty$). There are two takeaways from this example. First, joint production has an effect on aggregate TFP growth. Second, why does the formula omit the constant elasticity of transformation σ ? Given we observe prices, we don't need to estimate σ. The product-level price response encompasses all the required information about the joint production structure, satisfying assumptions 1 and 2.

4 Data and estimation

Our analysis relies on a dataset that covers the universe of formal firms operating in Chile from 2016 to 2022. This data is sourced from the Chilean Internal Revenue Service (Servicio de Impuestos Internos, SII). The Chilean tax system requires all formal firms to declare their invoices for transactions with other firms. This mandate provides us with detailed information on every product, quantity, and price traded between formal Chilean firms. Additionally, we have access to tax accounting declarations, which furnish monthly data on each firm's revenue and input expenditures, including capital and labor costs. We utilize four distinct sources from SII. A key advantage of the SII data is the use of unique identifiers for firms and workers, enabling the merging of individual and firm data across datasets.

The first source used is the value-added tax form, including gross monthly firm sales, materials expenditures, and investment.

Second, the SII provides information from a matched employer-employee census of Chilean firms from 2005. Specifically, firms must report their employee forms that record all firms' payments to individual workers: the sum of taxable wages, overtime, bonuses, and any other labor earnings for each fiscal year. Since all legal firms must report to the SII, the data covers the total labor force with a formal wage contract, representing roughly 65% of employment in Chile. For any given month, it is possible to identify the employment status of an individual worker, a measure of her average monthly labor income in that year, and a monthly measure of total employment and the distribution of average monthly earnings within the firm.

Third, data from the income tax form gathers yearly information on all sources of income and expenses of a firm. This form allows for the computing of every individual's actual tax payments for each year. Even though details on sales and employment are available on this form, we use only data on capital stock for each firm and year to build perpetual inventories using data from the monthly F22 form. The user cost of capital is obtained by multiplying nominal capital stock by the real rental rate of capital. The real rental rate of capital is built using publicly available data. We use the 10-year government bond interest rate minus expected inflation plus the external financing premium. Also, we use the capital depreciation rate from the LA-Klems database.

The fourth source comprises electronic tax documents, providing information on each product, including its price and quantity, traded domestically or internationally with at least one Chilean firm participant from 2016 onward. We will use only domestic transactions. We observe the universe of firm-to-firm transactions and the firm's total sales (which include both firm-to-firm and firm-to-consumer sales). We compute firm-specific product shares for the firm-to-firm universe and assume that their distributions are equivalent to firm-to-consumer transactions to recover the complete distribution of firm sales by product. Each firm-to-firm transaction reports a "detail" column that records the name of each product transacted.

Building on the data cleaning process described in Chapter 2, we further process the data to construct product code-level output and intermediate products input price indices for each firm using standard Tornqvist indices. To facilitate comparison between firms, we aggregate products into a 290 product code identifier. This aggregation allows us to estimate product production functions that use the same product across firms.

4.1 Data Cleaning and Implementation Strategy

The data processing begins with applying filters to the raw data to obtain the final database for empirical analysis. We define a firm as a taxpayer with a tax ID, positive sales, positive materials, positive wage bill, and capital for any given year. We exclude firms that hire less than two employees a year or have capital valued below US$20 in a year. To mitigate measurement error, all variables are winsorized at the 1% and 99% levels.

We construct product code-level output and intermediate products input price indices for each firm using standard Tornqvist indices. We selected 2016 as the base year for price indices because it was the first year in which we observed prices for firm-to-firm transactions. This method is widely recognized for estimating aggregate production functions at the firm or plant level when price data is accessible. To address the challenge of product aggregation (from around 15 million products to 290 product codes), crosswalks developed at the Central Bank of Chile are used (Acevedo et al. (2023)). We create aggregated product-level quantity produced and material usage indices. This process involves matching product descriptions and characteristics to ensure consistency across firms and over time. We report the aggregate statistics for the sample in Table 6 on the ratio of product-level sales to cost-based Domar weights and markups. The first two can be observed directly from the data without imputation. On the other hand, the distortion of the product itself, i.e., the markup, needs to be estimated. For markup estimation, we rely on the work by Dhyne et al. (2022), which developed an estimation method for multi-product production functions assuming joint production. This approach allows to capture the inefficiencies of production factors allocation within a firm. It captures the effect of how production changes of one product affect the production of other products within a firm. [7]

To make the implementation feasible, we allow firms to produce at most 5 of the 290 available product codes. We restrict product codes to account for at least 20% of the firm's total sales. All other products that represent less than 20% of the firm's total sales are grouped into a composite product that combines all the remaining products.

To construct the network distortion, we need information on the ratio of product-level sales to cost-based Domar weights and markups. The former two can be computed directly from the data without any parametric assumption or imputation. On the other hand, the distortion of the product itself, markups, needs to be estimated. For the markup estimation, we rely on the seminal work by Dhyne et al. (2022), which developed an estimation method for multi-product production functions assuming joint production. This approach allows us to capture the inefficiencies of production factor allocations within a firm when the production changes of one product affect the production of other products

[7] We cannot use the literature workhorse markup estimation strategy, De Loecker and Warzynski (2012) for multi-product markup. While we could use De Loecker et al. (2016) strategy on multi-product firms markups, that work relies on non-joint production setups, where each product production process is independent of other products within a firm. The latter approach is not that appealing because it omits the inefficiencies arising from the production process of one product, affecting the production process of other products within a firm.

of the same firm.[8]

4.2 Construction of Sufficient Statistics

To implement the growth accounting framework that includes the multi-product channel, we need to measure five distinct objects: (1) product-level cost-based Domar weights $\tilde{\boldsymbol{\lambda}}$, (2) product-firm level price indices, (3) product-level markups μ, (4) network distortion, and (5) aggregate objects. We will now discuss each of these.

4.2.1 Product-Level Cost-Based Domar Weights

The product cost-based Domar weights can be calculated using the following equation:

$$\tilde{\boldsymbol{\lambda}}' \equiv \boldsymbol{b}'\tilde{\Psi} = \boldsymbol{b}' + \boldsymbol{b}'\tilde{\Omega} + \boldsymbol{b}'\tilde{\Omega}^2 + \dots$$

To compute these weights, we need to measure two components: value-added shares ($\boldsymbol{b}$) and the input-output matrix ($\tilde{\Omega}$). We measure these objects directly from the data.

Final expenditure shares ($\boldsymbol{b}$) are represented by a vector of dimension $(\mathcal{N}\mathcal{G} + \mathcal{F}) \times 1$, where $\mathcal{N}$ is the number of firms, $\mathcal{G}$ is the number of products, and $\mathcal{F}$ is the number of factors. The first $\mathcal{N}\mathcal{G}$ entries are calculated as the residual between a firm product's total sales and its intermediate sales to other firms, which we measure from the firm-to-firm data. This approach provides a theory-consistent measure of final expenditures. The final $\mathcal{F}$ entries are set to zero, as households do not directly purchase factors. We construct the input-output matrix $\tilde{\Omega}$ at the product-firm level using firm-to-firm records and factor expenditures.

Specifically, we compute an annual cost-based input-output matrix by product. The denominator of each element (indexed by ig, jg') is calculated by summing a firm's purchases from all its suppliers, its wage bill, and its capital multiplied by the relevant user cost rental rate of capital. The last two elements of the matrix have wage bills and capital expenditures as their numerators.

The resulting $\tilde{\Omega}$ is a $(\mathcal{N}\mathcal{G} + 2) \times (\mathcal{N}\mathcal{G} + 2)$ matrix can be expressed as:

[8] We keep a record of product-specific prices and quantities to build price indices of the composite product.

$$\tilde{\Omega} = \left[\begin{array}{ccc|cc} \tilde{\Omega}_{11,11} & \cdots & \tilde{\Omega}_{11,\mathcal{NG}} & \tilde{\Omega}_{11,\mathcal{NG}+1} & \tilde{\Omega}_{11,\mathcal{NG}+2} \\ & \ddots & & & \\ \tilde{\Omega}_{\mathcal{NG},11} & & \tilde{\Omega}_{\mathcal{NG},\mathcal{NG}} & \tilde{\Omega}_{\mathcal{NG},\mathcal{NG}+1} & \tilde{\Omega}_{\mathcal{NG},\mathcal{NG}+2} \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right]$$

Based on the separability assumption, the same expenditure share applies to all products g that firm i produces. The expressions for $\tilde{\Omega}_{ig,jg'}$ and $\tilde{\Omega}_{ig,f}$ are as follows:

$$\tilde{\Omega}_{ig,jg'} = \frac{p_{jg'} x_{i,jg'}}{\sum_{j,p} p_{jg'} x_{i,jg'} + \sum_f w_f L_{if}}, \quad \tilde{\Omega}_{ig,f} = \frac{w_f L_{if}}{\sum_{j,p} p_{jg'} x_{i,jg'} + \sum_f w_f L_{if}}$$

Since factors do not require inputs, the last row of the matrix is zero.

After calculating the product-level cost-based Domar weights, we sum them for the same firms to compute the firm-level cost-based Domar weights and their shares. These will be used as inputs for Proposition 1.

$$\tilde{\lambda}_i = \sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig}, \quad s_{ig} = \frac{\tilde{\lambda}_{ig}}{\tilde{\lambda}_i}$$

4.2.2 Product-Firm Level Price Indices

We observe prices for each transaction and aggregate them into the 290 product categories we use. We construct two types of price indices: output price indices and input price indices. We compute firm-product-specific annual price indices for the output price index, which serves as an input to sufficient statistics and is used to deflate product output for production function estimation. The original data are at the "detail" product level, which we aggregate to a Tornqvist index for each 290 product category owned by the firm. Specifically, we construct the following price index:

$$\Delta \log P_{igt} = \sum_{d \in g} \frac{s_{idt} + s_{idt-1}}{2} \Delta \log P_{idt}$$

where d is the detailed category belonging to the upper product category (290 product codes), $\Delta \log P_{idt}$ is the price change, and s_{idt} is the share at time t in the continuing products in category g. We construct our price index using 2016, the starting year of the data, as the base year. We also construct an input price index, which is used to deflate material costs for production function estimation. We define one aggregate index per firm since aggregate materials are used as inputs in production function estimation. The construction method is the same as for the output price index.

4.2.3 Product-Level Markups

We estimate product-level markups using the production approach based on Dhyne et al. (2022). This method extends Ackerberg et al. (2015) production function estimation technique to a multi-product setting. It accounts for joint production, where firms simultaneously use common inputs to produce multiple products. The approach relies on cost minimization principles to identify unobserved marginal costs for each firm's product. We employ a Cobb-Douglas production function with three factors: capital, labor, and materials. Our results show a product-level markup median of 1.22. Please refer to Appendix D for a detailed explanation of the methodology.

4.2.4 Network distortion

To construct the network distortion measure, we require product cost-based Domar weights, product sales shares, and product markups:

$$\Gamma_{ig} \equiv \underbrace{\frac{\tilde{\lambda}_{ig}}{salesshare_{ig}}}_{\text{Downstream distortion}} \times \underbrace{\mu_{ig}}_{\text{own markup}},$$

While the first two components are directly observable in our data, the markup requires estimation. As discussed in Section 3, the ratio of cost-based Domar weights to sales share represents the cumulative distortion accumulated downstream of a product. The network distortion, Γ, is an essential input for constructing the multi-product term, and it is important to understand whether this variation arises from downstream distortions or from a product's markup.

Table 5: Variance decomposition of $\log \Gamma$

Year	Downstream distortions	Own markup	Covariance
2016	103.3%	0.6%	-3.9%
2017	102.3%	0.7%	-3.0%
2018	102.5%	0.6%	-3.1%
2019	102.8%	0.6%	-3.5%
2020	103.2%	0.7%	-3.8%
2021	103.7%	0.6%	-4.3%
2022	104.9%	0.7%	-5.6%

Notes: We compute the variance decomposition of the logarithm of $Gamma_{ig} \equiv \frac{\tilde{\lambda}_{ig}}{salesshare_{ig}} \times \mu_{ig}$ for each year. $Var\left(\log \Gamma_{ig}\right) = Var\left(\log\left(\tilde{\lambda}_{ig}/salesshare_{ig}\right)\right) + Var\left(\log \mu_{ig}\right) + 2Cov\left(\log\left(\tilde{\lambda}_{ig}/salesshare_{ig}\right), \log \mu_{ig}\right)$. The first term on the right-hand side is the variance of downstream distortions, the second term is the variance of their own markup, and the last is the covariance of both. We report the percentage of each term on the right-hand side that explains the total variance.

Table 5 presents the variance decomposition of Γ by year. The results show that most of the variation in Γ stems from downstream distortions, with minimal contribution from the product's markup. This finding is not surprising, given that downstream distortions represent cumulative wedges throughout the downstream supply chain of the entire economy, whereas μ simply represents a product's own markup. This result also implies that the downstream distortions faced by each pair of firms and products are highly heterogeneous when considering product- and firm-level production networks.

This latter anticipates that when we apply Proposition 1 to the data, the network distortion (the multi-product term using Γ as input) will be less sensitive to markup estimates. Indeed, in the following section, we present results using Γ without the markups and demonstrate the robustness of our findings when we approximate the model around a state with an initial common markup within the firm.

Ranking of Network Distortions

The analysis reveals that network distortions primarily represent accumulated downstream distortions rather than product-level markups. To better understand which products face relatively greater downstream distortion, we ranked products by their network distortion. Below, we describe and discuss the major product categories. The complete list of the top and bottom 30 items is provided in Appendix C.

The product categories with the greatest (downstream) distortions are mainly business services product categories. For example, insurance brokerage services top the list, followed by employment services (recruitment and supply), electricity distribution to businesses, and postal and courier services. These products are usually upstream inputs used in production by other firms, suggesting that their size is too small as distortions accumulate through the supply chain before they reach final demand. An important exception is tobacco, which is a product close to final demand but is ranked high (15th) because of the large wedge that has been accumulated (59.7% tax rate vs 19% VAT tax for other products).

Conversely, the least distorted products include cakes, beer, pet food, personal services such as hospitals, and Chile's main export industries: minerals (copper, silver, molybdenum). These products are, in common, downstream products close to the final demand (for Chile). As a result, the number of supply chains that reach the final consumer is relatively small, and inefficiencies are relatively less likely to accumulate.

4.2.5 Aggregate Objects

In addition to product cost-based Domar weights, markups, and product distortions, we need to measure aggregate objects to implement the sufficient statistics presented in Proposition 1. In particular, Y, L, K, Λ_L, and Λ_K, denote aggregate value-added, employment, capital, and factor shares, respectively. We measure Y, L, and K as the sum of value added, employment, and capital, respectively, of all firms in the economy. Factor shares of GDP, Λ_L and Λ_K are measured as total compensation and capital with user cost of capital divided by GDP. Real GDP is calculated by deflating GDP with the official GDP deflator.

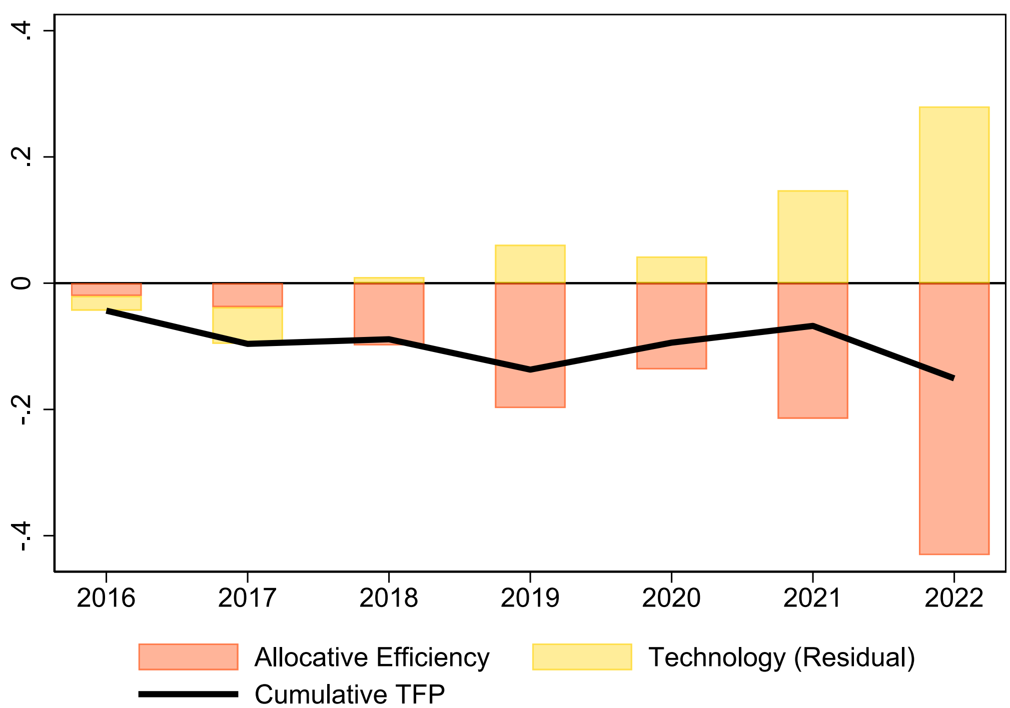
5 Application: Decomposing Aggregate TFP Growth

In this section, we apply Proposition 1 to analyze aggregate TFP growth for the Chilean economy. Our analysis covers the period from 2016 to 2022, during which Chile's aggregate TFP stagnated, and decreased at the margin. This productivity trend aligns with the pattern of productivity stagnation observed in chile with different computation methods

[9] and trends observed in different countries globally.

We begin our analysis with a standard assumption in the literature: the single-product firm model. This approach disregards multi-product firms and implements growth accounting without considering the multi-product term. Figure 5 illustrates the decomposition of cumulative changes in aggregate TFP from 2016 to 2022 under this assumption. As shown in Figure 5, the allocative efficiency term (in red) declined over this period. This suggests that high-markup firms have contracted further, resulting in a negative reallocation effect. However, the contribution of the allocative efficiency is larger than the technology (residual) one, particularly during the COVID-19 pandemic and the subsequent high inflation period. To rationalize this, the technical terms measured as residuals must increase by around 20%.

Figure 5: Cumulative TFP growth decomposition: ignoring multi-product term



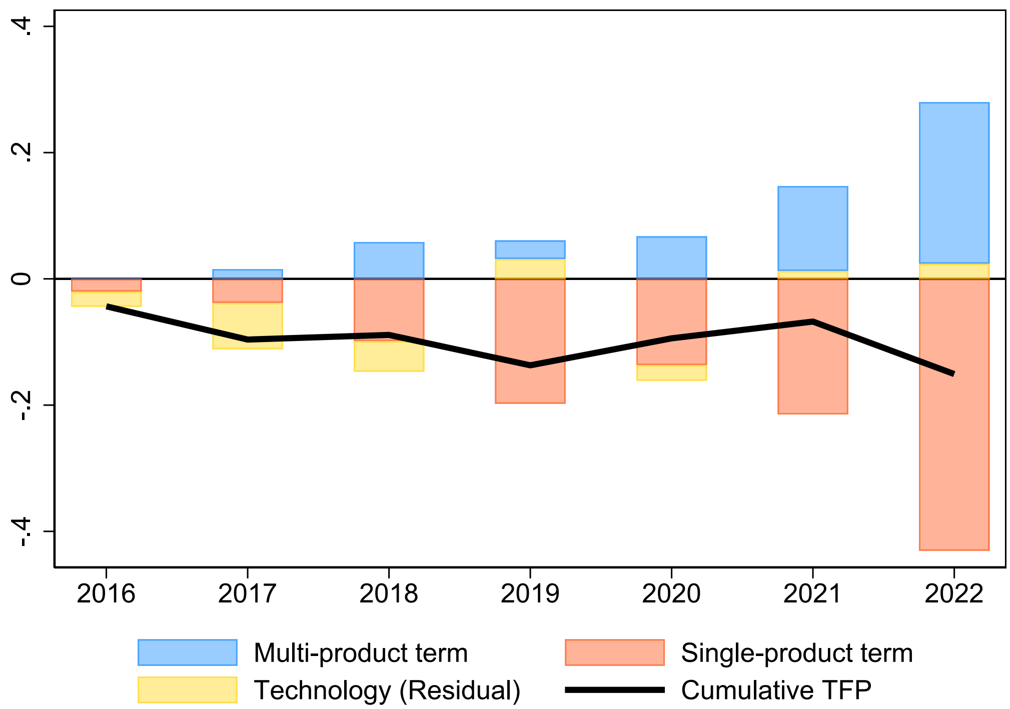
Notes: This Figure shows the cumulative change calculated by applying Corollary 1 repeatedly each year. Technology (residual) is calculated by subtracting allocative efficiency from TFP growth.

Next, Figure 6 incorporates the sufficient statistics representing the multi-product term. The multi-product term makes a smaller contribution during normal times (pre-

[9] CNEP (2023)

2019) compared to the single-product term. However, after 2020, it made a substantial upward contribution during the COVID-19 pandemic and the subsequent high inflation period. This reduces the magnitude of the technology (residual) observed in Figure 5. In other words, the multi-product term, together with the single-product misallocation term, accounts for aggregate TFP movements during the COVID-19 pandemic and the following high inflation period.

Figure 6: Cumulative TFP growth Decomposition with multi-product term



Notes: This Figure shows the cumulative log change calculated by repeatedly applying the equation from Proposition 1 each year. Technology (residual) is calculated by subtracting allocative efficiency from aggregate TFP growth.

Reallocation effects considering joint production explain 86% [10] of the observed aggregate TFP growth. Ignoring joint production leads to overestimating resource misallocation. This can be interpreted as multi-product firms facing constraints in adjusting their product portfolio, limiting the scope for reallocation within the network.

The contribution of joint production forces increases during economic disruptions. Af-

[10] We first sum the blue and red bars, which we call allocative efficiency, and we compute its contribution to the total variance of cumulative aggregate TFP growth.

ter the COVID-19 pandemic and subsequent high inflation period, the joint production term of allocative efficiency increased. This suggests that product-level demand composition changed in response to economic disruptions, requiring firms to adjust their product portfolios. When firms engage in joint production they can't fully adjust their product portfolios in the same way that non-joint production firms do. Joint production firms face technological constraints when adjusting product portfolios using common inputs.

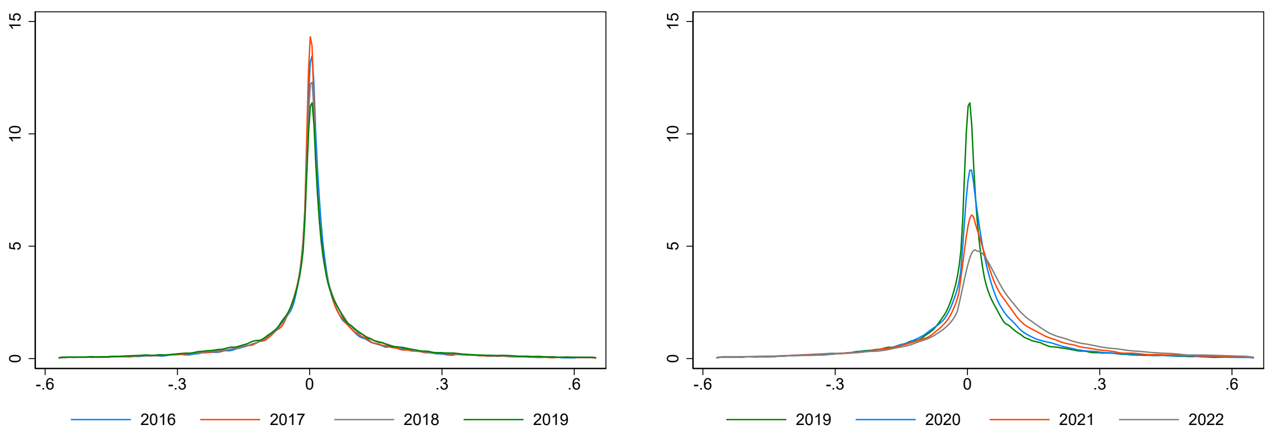
Ignoring technical constraints from joint production and assuming firms can freely adjust their product portfolios—as in single-product models—leads to overestimating reallocation forces. These results suggest that multi-product firms become particularly important in understanding aggregate TFP dynamics during periods of economic downturn.

The granularity of the data allows us to track the distributional changes of joint production (Multi-product term) limiting resource reallocation forces. Since the covariance degenerates to zero under the single-product firm assumption, the dispersion of covariance implies joint production forces are active. We find that these distributions vary from period to period. Figure 7a plots the distribution for pre-COVID-19, (2016-2019), which is symmetric around 0, with small differences from year to year.

In contrast, Figure 7b shows the distribution after COVID-19. This distribution shifts to the right from year to year, resulting in a right-skewed distribution. This suggests that the increase in joint production forces (multi-product term) is not driven by a few specific firms.

Figure 7: $Cov_{\boldsymbol{s}_i}\left(d\log \boldsymbol{p}_i, \frac{1}{\gamma_i}\right)$ distributions by year

(a) 2016 - 2019 (b) 2019 - 2022

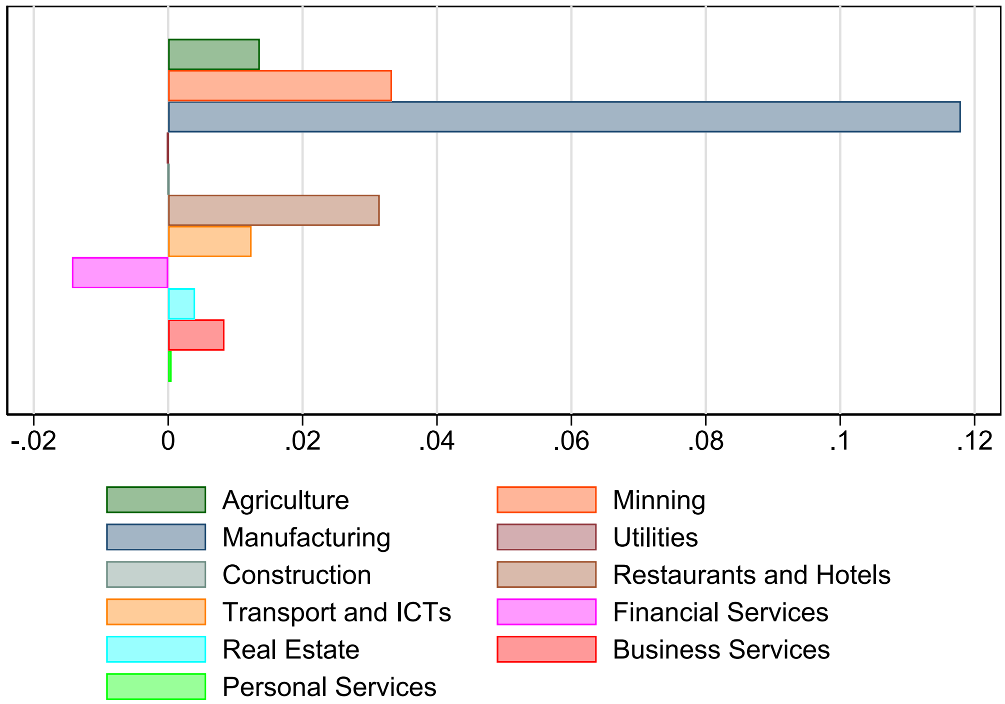


Notes: These Figures plot the distribution of firm-level $Cov_{s_i}\left(d\log \boldsymbol{p}_i, \frac{1}{\gamma_i}\right)$ for each year.

To explore this further, Figure 8 decomposes the joint production forces through the cumulative multi-product channel by 11 product categories. We categorize firms based on the 1-digit category of their main product (defined as the product with the largest sales within a given firm) and plot the cumulative changes from 2019 to 2022 for each category. The aggregate contribution by product categories of the firm's main product through the multi-product channel is given by:

$$\text{Contribution of firm with product } c = \sum_{i\in N_C} \tilde{\lambda}_i Cov_{s_i}\left(d\log \boldsymbol{p}_i, \frac{1}{\gamma_i}\right)$$

Figure 8: Multi-product term by category from 2019 to 2022



Notes: This Figure shows the cumulative changes of multi-product term by categories from 2019 to 2022.

The largest contribution is from the manufacturing product category. However, apart from construction and finance services, all other categories make a positive contribution.

6 Conclusion

This paper develops a theoretical framework to aggregate distortions in production networks with multi-product firms and assess their impact on aggregate TFP growth. We derive a non-parametric sufficient statistic to describe allocative efficiency in the presence of multi-product firms engaging in joint production.

We apply the framework to a comprehensive Chilean firm-to-firm transaction database. Reallocation effects considering joint production explain 86% of the observed aggregate TFP growth. Ignoring joint production leads to overestimating resource misallocation. This can be interpreted as multi-product firms facing constraints in adjusting their product portfolios, limiting the scope for reallocation within production networks.

References

Acevedo, P., Luttini, E., Pizarro, M., and Quevedo, D. (2023). Invoices rather than surveys: using ML to nominal and real indices.

Ackerberg, D. A., Caves, K., and Frazer, G. (2015). Identification properties of recent production function estimators. *Econometrica: journal of the Econometric Society*, 83(6):2411–2451.

Almunia, M., Antràs, P., Lopez-Rodriguez, D., and Morales, E. (2021). Venting out: Exports during a domestic slump. *The American economic review*, 111(11):3611–3662.

Bai, X., Fernández-Villaverde, J., Li, Y., and Zanetti, F. (2024). The causal effects of global supply chain disruptions on macroeconomic outcomes: Evidence and theory. *SSRN Electronic Journal*.

Baqaee, D., Burstein, A., Duprez, C., and Farhi, E. (2023). *Supplier Churn and Growth: A Micro-to-Macro Analysis*. National Bureau of Economic Research.

Baqaee, D. R. and Farhi, E. (2020). Productivity and misallocation in general equilibrium. *The quarterly journal of economics*, 135(1):105–163.

Basu, S. and Fernald, J. G. (2002). Aggregate productivity and aggregate technology. *European economic review*, 46(6):963–991.

Bernard, A. B., Redding, S. J., and Schott, P. K. (2010). Multiple-product firms and product switching. *The American economic review*, 100(1):70–97.

Bernard, A. B., Redding, S. J., and Schott, P. K. (2011). Multiproduct firms and trade liberalization. *The Quarterly Journal of Economics*, 126(3):1271–1318.

Bigio, S. and La'O, J. (2020). Distortions in production networks*. *The quarterly journal of economics*, 135(4):2187–2253.

Boehm, C. E., Flaaen, A., and Pandalai-Nayar, N. (2019). Input linkages and the transmission of shocks: Firm-level evidence from the 2011 tōhoku earthquake. *The review of economics and statistics*, 101(1):60–75.

Boehm, J., Dhingra, S., and Morrow, J. (2022). The comparative advantage of firms. *The journal of political economy*, 130(12):3025–3100.

Boehm, J. and Oberfield, E. (2023). Growth and the fragmentation of production.

Burstein, A., Cravino, J., and Rojas, M. (2024). Input price dispersion across buyers and misallocation.

Cairncross, J. and Morrow, P. (2023). Multi-product markups.

Carrillo, P., Donaldson, D., Pomeranz, D., and Singhal, M. (2023). *Misallocation in Firm Production: A Nonparametric Analysis Using Procurement Lotteries*. National Bureau of

Economic Research.

Carvalho, V. M., Nirei, M., Saito, Y. U., and Tahbaz-Salehi, A. (2020). Supply chain disruptions: Evidence from the great east japan earthquake*. *The quarterly journal of economics*, 136(2):1255–1321.

CNEP, C. N. E. . P. C. (2023). Informe anual de productividad 2023.

De Loecker, J., Goldberg, P. K., Khandelwal, A. K., and Pavcnik, N. (2016). Prices, markups, and trade reform. *Econometrica: journal of the Econometric Society*, 84(2):445–510.

De Loecker, J. and Warzynski, F. (2012). Markups and firm-level export status. *The American economic review*, 102(6):2437–2471.

Dhyne, E., Petrin, A., Smeets, V., and Warzynski, F. (2017). Multi product firms, import competition, and the evolution of firm-product technical efficiencies. (23637).

Dhyne, E., Petrin, A., Smeets, V., and Warzynski, F. (2022). Theory for extending single-product production function estimation to multi-product settings. (30784).

Diewert, W. E. (1971). An application of the shephard duality theorem: A generalized leontief production function. *The journal of political economy*, 79(3):481–507.

Diewert, W. E. (1973). Functional forms for profit and transformation functions. *Journal of Economic Theory*.

Ding, X. (2023). *Industry Linkages from Joint Production*. U.S. Census Bureau, Center for Economic Studies.

Fujiy, B. C., Ghose, D., and Khanna, G. (2022). Production networks and firm-level elasticities of substitution.

Giroud, X. and Mueller, H. M. (2019). Firms' internal networks and local economic shocks. *The American economic review*, 109(10):3617–3649.

Hall, R. (1988). The relation between price and marginal cost in U.S. industry. *The journal of political economy*, 96(5):921–947.

Hall, R. E. (1973). The specification of technology with several kinds of output. *The journal of political economy*, 81(4):878–892.

Hottman, C. J., Redding, S. J., and Weinstein, D. E. (2016). Quantifying the sources of firm heterogeneity. *The Quarterly Journal of Economics*, 131(3):1291–1364.

Hsieh, C.-T. and Klenow, P. J. (2009). Misallocation and manufacturing TFP in china and india. *The quarterly journal of economics*, 124(4):1403–1448.

Hulten, C. R. (1978). Growth accounting with intermediate inputs. *The Review of economic studies*, 45(3):511–518.

Kikkawa, A. K. (2022). Imperfect competition in firm-to-firm trade. *Journal of the European Economic Association*, 20(5):1933–1970.

Lau, L. J. (1972). Profit functions of technologies with multiple inputs and outputs. *The review of economics and statistics*, 54(3):281–289.

Mayer, T., Melitz, M. J., and Ottaviano, G. I. P. (2014). Market size, competition, and the product mix of exporters. *The American economic review*, 104(2):495–536.

Mayer, T., Melitz, M. J., and Ottaviano, G. I. P. (2021). Product mix and firm productivity responses to trade competition. *The review of economics and statistics*, 103(5):874–891.

Osotimehin, S. and Popov, L. (2023). Misallocation and intersectoral linkages. *Review of economic dynamics*.

Petrin, A. and Levinsohn, J. (2012). Measuring aggregate productivity growth using plant-level data. *The Rand journal of economics*, 43(4):705–725.

Powell, A. A. and Gruen, F. H. G. (1968). The constant elasticity of transformation production frontier and linear supply system. *International economic review*, 9(3):315–328.

Restuccia, D. and Rogerson, R. (2008). Policy distortions and aggregate productivity with heterogeneous establishments. *Review of economic dynamics*, 11(4):707–720.

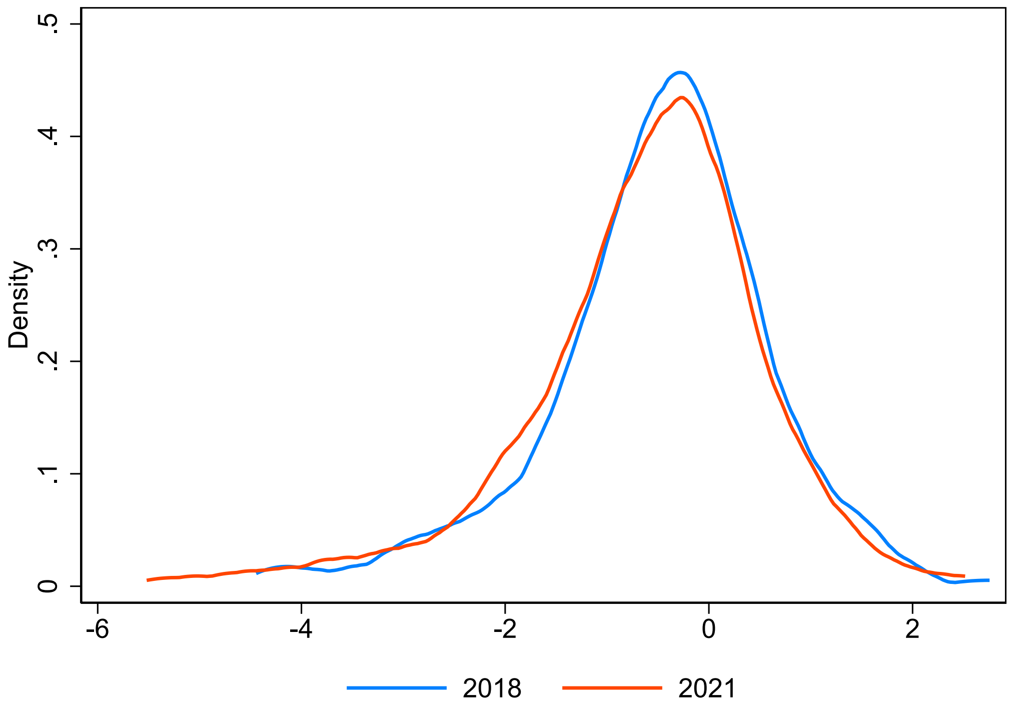
Solow, R. M. (1957). Technical change and the aggregate production function.

Valmari, N. (2023). Estimating production functions of multiproduct firms. *The Review of economic studies*.

Appendix

A Additional Figures and Tables

Figure 9: Distribution of Γ_{ig} by year



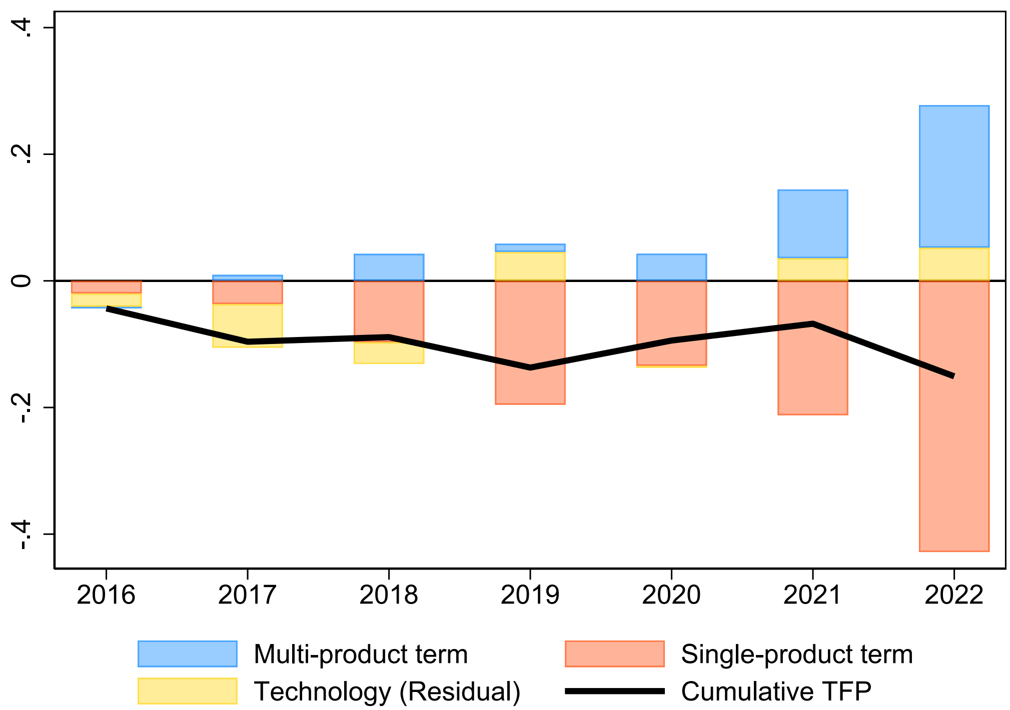
Note: This figure plots the value of the median for each CUP unit of log Γ. The Γ is normalized to the median value of 0.

Growth accounting results assuming that markups within firms are equal in initial equilibrium

We show the results when the equilibrium is approximated around $\mu_{ig} = \mu_i$. Note that the level of μ_i does not affect the results since Γ is normalized for covariance.

$$\Gamma_{ig} \equiv \underbrace{\frac{\tilde{\lambda}_{ig}}{salesshare_{ig}}}_{\text{Downstream distortion}} \times \underbrace{\mu_i}_{\text{own markup}},$$

Figure 10: Cumulative TFP Decomposition with Multi-Product Term



Note: This Figure shows the cumulative log change calculated by repeatedly applying the equation from Proposition 1 each year. Technology (residual) is calculated by subtracting allocative efficiency from TFP.

Table 6: Aggregate firm-level statistics

Year	Count	Sales	Wagebill	Employment
2016	110,451	262,506	40,260	4,242,555
2017	114,480	277,960	43,691	4,349,248
2018	115,916	330,486	44,688	4,349,454
2019	116,706	336,386	47,299	4,425,780
2020	102,306	310,317	44,053	3,935,883
2021	105,651	376,220	51,642	4,166,838
2022	105,032	454,818	59,148	4,266,972

Note: Count stands by the number of firms, while sales and wage bill are yearly aggregates expressed in million pesos. Employment represents the headcount of yearly workers included in the sample.

B Comparison with Baqaee and Farhi (2020) with simple example with non-joint production

We apply Baqaee and Farhi (2020)'s formula to Example 3.5 to show that the same results are obtained when the production technology is non-joint. In this example, we model the production technology of firm 1 as non-joint production.

The production technology for Example 3.5 is as follows. Define the production technology of firm 1 as follows (productivity normalized to 1):

$$q_{11} = L_{11}, \quad q_{12} = L_{12},$$

Firm 2 uses linear technology and sells product 2 as is:

$$q_{22} = q_{12}$$

To apply Baqaee and Farhi (2020)'s formula, unlike ours, we treat all products as if they were separate firms. Therefore, we have

$$\begin{aligned} \underbrace{-\sum_{i \in \mathcal{N}} \sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig} d \log \mu_{ig}}_{\text{product level Markup}} - \underbrace{d \log \Lambda_f}_{\text{Aggrregate Labor Shares}} &= -\frac{1}{2}(-\epsilon) - \left(-\frac{1}{\mu+1}\right)(-\epsilon) \\ &= \left(\frac{\mu - 1}{2(1+\mu)}\right)\epsilon \end{aligned}$$

Therefore,

$$\Delta \log TFP = \left(\frac{\mu - 1}{2(1+\mu)}\right)\epsilon > 0$$

The same results were obtained as in Example 3.5. Importantly, this identity holds only when all production technologies are non-joint.

C Product distortions

Table 7: The 30 most distorted products

Ranking	Description
1	Insurance brokerage services
2	Other services
3	Passenger air transport services
4	Wholesale trade intermediary services
5	Electricity distribution and other related services
6	Investigation and security services
7	Airport services
8	Radio and open TV broadcast services
9	Wastewater treatment services
10	Online content services
11	Cleaning services
12	Liquefied Natural Gas
13	Employment services (placement and supply)
14	Postal and courier services
15	Tobacco
16	Paper and cardboard containers, paper or cardboard for recycling
17	Other IT services
18	News agency services
19	Margarine and similar preparations, other residues and waste from fats
20	General insurance
21	Other rubber products
22	Other auxiliary and complementary services for education services
23	Other goods or services not classified elsewhere
24	Long-distance passenger transport services
25	Gas distribution services and other related services
26	Some other product
27	Maritime passenger transport services
28	Research and development services
29	Repair and installation of machinery and equipment, except for the textile industry
30	Database software licensing services

Note: For 2018, products are ranked using the network distortion medians for CUP's product categories, and products with the top 30 distortion sizes are reported.

Table 8: The 30 Least distorted product

Ranking	Description
1	Molybdenum minerals and their concentrates
2	Other non-metallic minerals
3	Gaseous natural gas
4	Crude oil
5	Mining works
6	Unrefined copper, ashes, residues and wastes of copper
7	Silver
8	Public administration and defense services; compulsory social security plans
9	Pet food
10	Bird food
11	Fish meal, crustacean, mollusk and other aquatic invertebrate meal
12	Ammonium nitrate
13	Lease services with or without purchase option
14	Bread
15	Veterinary services
16	Poultry meat and edible offal
17	Integrated telecommunications services (packs)
18	Fuel oil
19	Beers
20	Life insurance
21	Cakes, cakes and cookies
22	Hake
23	Consultancy and post services
24	Copper minerals and their concentrates
25	Public hospital services
26	Social and association services
27	Petroleum gas and other gaseous hydrocarbons, except natural gas
28	Fish oil
29	Mining exploration and evaluation services
30	Housing services

Note: For 2018, products are ranked using the network distortion medians for CUP's product categories, and products with the top 30 distortion sizes are reported.

D Detailed Methodology for Product-Level Markup Estimation

We estimate product-level markups following the production approach based on Dhyne et al. (2022). In a joint production setup, firms use common inputs to produce a product portfolio, meaning that some inputs may simultaneously be used to produce multiple products. They proposed an Ackerberg et al. (2015) like production function estimation method based on Diewert (1973)'s production set. The following is an overview of Dhyne et al. (2022)'s methodology.

A firm has production possibilities set, V, that consists of a set of feasible inputs $\boldsymbol{x} = (x_1, ..., x_M)$ and outputs of the product,$\boldsymbol{q} = (q_1, ..., q_G)$. For any (q_g, x) the transformation function is defined as

$$q_g^* = f_g\left(q_g, x\right) \equiv \max\left\{q_g | \left(q_g, q_{-g}, x\right) \in V\right\}$$

To identify the unobserved marginal cost for each firm's product, we rely on (variable) cost minimization. Firms have $N-1$ freely variable inputs and one fixed input, capital (K), so the problem that a firm faces to minimize its variables cost to produce its output vector q_i^* given the input prices vector $\boldsymbol{p_x} = (p_{x1}, ..., p_{xM})$ and unobserved productivity for products, $\boldsymbol{\omega} = (\omega_1, ..., \omega_G)$.

Defining the Lagrangian multiplier of the cost minimization problem, mc_g, as the marginal cost of product g, the first order condition for every optimal input demand yield:

$$p_m = mc_g \frac{\partial f(q^*_{-g}, x, K, \boldsymbol{\omega})}{\partial x_m} \quad \forall m = 1, .., M,$$

It is possible to solve for product g marginal cost as the expenditure on production input m divided by its output elasticity (β_n^g) times product g production:

$$mc_g = \frac{p_m}{\frac{\partial f(q^*_{-g}, x, K, \omega)}{\partial x_m}} = \frac{p_m x_m^*}{\beta_m^g q_g^*},$$

Multiplying the marginal cost expression by $\frac{1}{p_g}$, where p_g is product g price, product g markup is given by:

$$\mu_g = \beta_m^g \frac{p_g q_g^*}{p_m x_m^*},$$

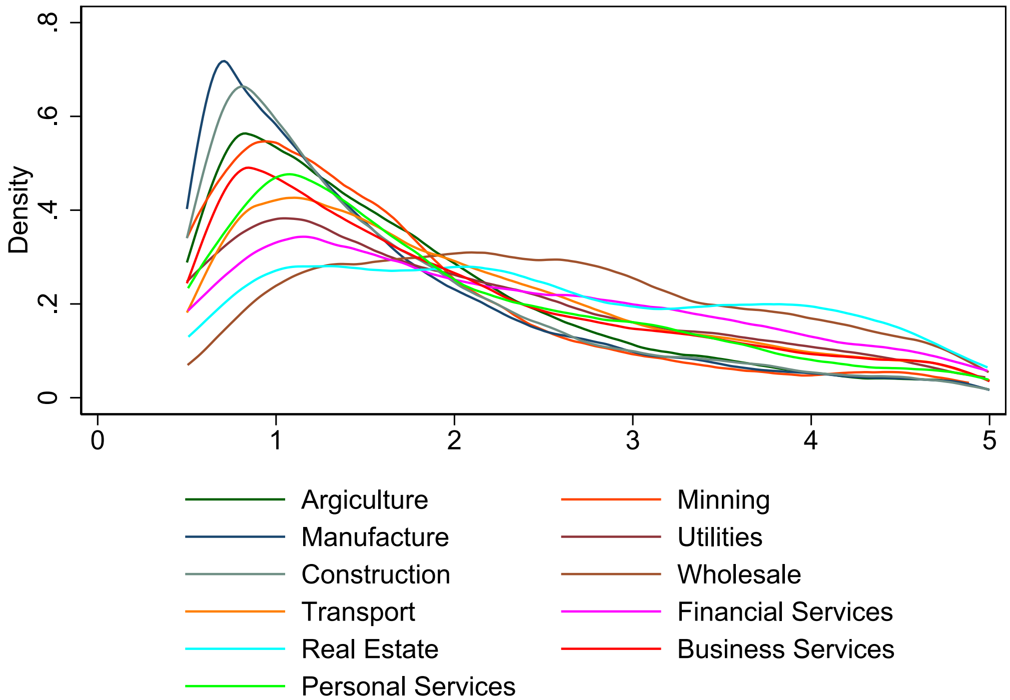
We use control functions for the unobserved productivity terms (i.e., Ackerberg et al. (2015)) to account for unobserved productivity with the difference of the need for instruments for q_{-g}; following Dhyne et al. (2022) we use lagged values of q_{-g}. We assume that firms use a Cobb-Douglas production function with three factors: (Capital K, Labor L, and Materials M). A multi-product firm will produce physical units of product g using the following production function:

$$\log q_{gt} = \beta_0^g + \beta_k^g \log k_t + \beta_l^g \log l_t + \beta_m^g \log m_t^j + \gamma_{-g}^g \log q_{-gt} + \omega_{gt}$$

We pool together products at one digit (12 aggregate product categories) and perform production function estimations separately for each category following ACF using a GMM estimator.

Product-level markup distribution concentrated around 1, with a 1.22 median. We remain agnostic about product-level markup interpretation. While the markup distribution offers valuable insights into diagnosing the presence of product market power, it can potentially lead to misleading conclusions regarding allocative efficiency. Markup distribution does not account for the significance of firms within production networks.

Figure 11: Product-level markup distribution in 2018



1

E Proofs

Proof of Example 1 in subsection 3.5:

Proof. First, we compute relative network distortions:

$$\gamma_{ig} \equiv \frac{\Gamma_{ig}}{\Gamma_i}$$

where the average distortion of firm i is defined as: $\Gamma_i \equiv \sum_g \tilde{\lambda}_{ig} / \sum_g \left(salesshare_{ig} / \mu_{ig} \right)$. So Γ_1 is

$$\Gamma_1 \equiv \frac{1}{\frac{1}{\mu} + \frac{1}{\mu^2}} 2 = \frac{2}{\frac{1}{\mu}\left(1 + \frac{1}{\mu}\right)}$$

$$\gamma_{11} = \frac{\mu^2}{\frac{2}{\frac{1}{\mu}\left(1+\frac{1}{\mu}\right)}} = \frac{(\mu + 1)}{2}$$

$$\gamma_{12} = \frac{\mu}{\frac{2}{\frac{1}{\mu}\left(1+\frac{1}{\mu}\right)}} = \frac{\mu\frac{1}{\mu}\left(1+\frac{1}{\mu}\right)}{2} = \frac{\left(1+\frac{1}{\mu}\right)}{2}$$

Thus,

$$\frac{1}{\gamma_{11}} = \frac{2}{\frac{1+\mu}{\mu}} = 2\frac{1}{(\mu+1)}$$

$$\frac{1}{\gamma_{12}} = \frac{2}{\frac{1+\mu}{\mu}} = 2\frac{\mu}{1+\mu}$$

Substitute them into covariance:

$$\begin{aligned}
Cov_{s_i}\left(d\log \boldsymbol{p}_i, \frac{1}{\gamma_i}\right) &= Cov_{\left[\frac{1}{2},\frac{1}{2}\right]}\left(\begin{bmatrix} -\epsilon \\ 0 \end{bmatrix}, \begin{bmatrix} 2\frac{1}{(\mu+1)} \\ 2\frac{\mu}{(\mu+1)} \end{bmatrix}\right) \\
&= \left(\frac{\mu-1}{2(1+\mu)}\right)\epsilon
\end{aligned}$$

Next, we compute the endogenous response of Labor share $d\log\Lambda$ to markup shock. By factor share identity, we know

$$\begin{aligned}
\Lambda_L &= 1 - \left(1 - \frac{1}{\mu}\right) - \frac{\alpha}{\mu}\left(1 - \frac{1}{\mu_{11}}\right) \\
\Lambda_L &= \frac{1}{\mu} - \frac{1}{2}\frac{1}{\mu}\left(1 - \frac{1}{\mu_{11}}\right) \\
&= \frac{1}{\mu}\frac{1}{2}\left(1 + \frac{1}{\mu_{11}}\right)
\end{aligned}$$

$$= \frac{1}{\mu}\frac{1}{2}\left(1 + \frac{1}{\mu_{11}}\right)$$

$$\log\Lambda = -\log\mu + \log\left(\frac{1}{\mu_{11}} + 1\right)$$

$$\begin{aligned}
\frac{d\log\Lambda}{d\log\mu_{11}} &= \frac{d\log\left(\frac{1}{\mu_{11}}+1\right)}{d\mu_{11}}\frac{d\mu_{11}}{d\log\mu_{11}} \\
&= \frac{-\frac{1}{\mu_{11}^2}}{\frac{1}{\mu}+1}\mu_{11}
\end{aligned}$$

$$= \frac{-\frac{1}{\mu_{11}}}{\frac{1+\mu}{\mu}}$$

Evaluate at $\mu_{11} = \mu$ gives

$$\frac{d \log \Lambda}{d \log \mu_{11}} = -\frac{1}{\mu + 1}$$

So

$$\begin{aligned} d \log \Lambda &= \frac{d \log \Lambda}{d \log \mu_{11}}(-\epsilon) \\ &= -\frac{1}{\mu + 1}(-\epsilon) \end{aligned}$$

And

$$\sum_i \tilde{\lambda}_i d \log \mu_i = \frac{1}{\mu + 1}(-\epsilon)$$

Therefore,

$$-d \log \Lambda - \sum_i \tilde{\lambda}_i d \log \mu_i = 0$$

□

Proof of Example 2 in subsection 3.5:

Proof. Pick product 2 to be a reference product for firm 1. Then, we know

$$d \log p_{11}/p_{12} = d \log \mu_{11}/\mu_{12} + \frac{1}{\sigma} d \log y_{11}/y_{12}$$

Using $d \log \lambda = d \log p + d \log y$

$$d \log p_{11}/p_{12} = \left(\frac{\sigma}{\sigma + 1}\right) d \log \mu_{11}/\mu_{12} + \frac{1}{\sigma + 1} d \log \lambda_{11}/\lambda_{12}$$

$d \log \mu_{11} = -\epsilon$, $d \log \mu_{12} = 0$, and $d \log \mu_{21} = 0$. By Cobb Douglas assumption, we know $d \log \lambda_{11}/\lambda_{12} = 0$. Thus we have

$$d \log p_{11}/p_{12} = -\left(\frac{\sigma}{\sigma + 1}\right)\epsilon$$

$$
\begin{aligned}
d\log TFP &= Cov_{\left[\frac{1}{2},\frac{1}{2}\right]}\left(\begin{bmatrix} d\log p_{11} \\ d\log p_{12} \end{bmatrix}, \begin{bmatrix} \frac{1}{\gamma_1} \\ \frac{1}{\gamma_2} \end{bmatrix}\right) \\
&= Cov_{\left[\frac{1}{2},\frac{1}{2}\right]}\left(\begin{bmatrix} d\log p_{11}/p_{12} \\ d\log p_{12}/p_{12} \end{bmatrix}, \begin{bmatrix} 2\frac{\mu}{1+\mu} \\ 2\frac{1}{(\mu+1)} \end{bmatrix}\right) \\
&= Cov_{\left[\frac{1}{2},\frac{1}{2}\right]}\left(\begin{bmatrix} -\left(\frac{\sigma}{\sigma+1}\right)\epsilon \\ 0 \end{bmatrix}, \begin{bmatrix} 2\frac{\mu}{1+\mu} \\ 2\frac{1}{(\mu+1)} \end{bmatrix}\right) \\
&= \left(\frac{\sigma}{\sigma+1}\right)\left(\frac{\mu-1}{2(1+\mu)}\right)\epsilon
\end{aligned}
$$

□

Proof of Proposition 1

Lemma 1. *Price equation with multi-product firms for some reference product r of firm i:*

$$
\frac{y_{ir}mc_{ir}}{C(y_i,p)}d\log p_{ir} = -d\log A_i/\mu_i + \underbrace{\sum_{j,k}\frac{p_{jg}x_{i,jg'}}{C(y_i,p)}d\log p_{jg'} + \sum_f \frac{w_f l_{if}}{C(y_i,p)}d\log w_f}_{\text{intermediate and factorprice}}
$$

$$
+\underbrace{\sum_{g\neq r}\left(-\frac{y_{ig}mc_{ig}}{C(y_i,p)}\right)d\log p_{ig}}_{\text{other product from the same firm}}
$$

Proof. By CRS, we know

$$
C_i(\boldsymbol{q}_i,\boldsymbol{p}_i) = \sum_g q_{ig}mc_{ig}
$$

Total derivative:

$$
RHS = \sum_g \frac{\partial\log\left(\sum q_{ig}mc_{ig}\right)}{\partial\log y_i}y_i d\log y_i + \sum_i \frac{\partial\log\left(\sum q_{ig}mc_{ig}\right)}{\partial\log mc_{ig}}mc_{ig}d\log mc_{ig}
$$

$$= \sum_g \frac{q_{ig} mc_{ig}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log q_{ig} + \sum_g \frac{q_{ig} mc_{ig}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log mc_{ig}$$

and

$$LHS = -d\log A_i + \sum_{j,g'} \frac{p_{jg'} x_{i,jg'}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log p_{jg'} + \sum_f \frac{w_f l_{i,f}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log w_f + \sum_g \frac{q_{ig} mc_{ig}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log q_{ig}$$

Hence,

$$\sum_g \frac{y_{ig} mc_{ig}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log mc_{ig} = -d\log A_i + \sum_{j,g'} \frac{p_{jg'} x_{i,jg'}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log p_{jg'} + \sum_f \frac{w_f L_{if}}{C_i(\boldsymbol{q}_i, \boldsymbol{p}_i)} d\log w_f \tag{5}$$

□

This equation is valid for any joint production that satisfies the CRS assumption. Due to the interdependence of output prices within firms, this equation alone cannot be used to determine prices, unlike in the case of single-product firms. Additional equations governing the relationship with the prices of other products are needed to pin down prices. For this reason, below, we derive an equation that connects the other prices within a firm.

Cost minimization with transformation function:

Assume the implicit function theorem can be applied so that we could pick some reference product r of firm i and explicitly functional form locally: $y_{ir} = f_{ir}(y_{i,-r}, \boldsymbol{x_i}, \boldsymbol{L_i})$ where $-r$ refer all $g \neq r$. $\boldsymbol{x_i}$ is the vector of intermediate products and $\boldsymbol{L_i}$ is the vector of factors.

Then, we could solve the cost-minimization problem:

$$C_i(\boldsymbol{y}_i, \boldsymbol{p}_i) = \sum_f w_{if} x_f + mc_{ir}(y_{ir} - f_{ir}(\boldsymbol{y}_{i,-r}, \boldsymbol{x}_i, \boldsymbol{L}_i))$$

By taking a derivative with respect to y_{ig} for $g \neq r$, we have

$$mc_{ig} = mc_{ir}\left(-\frac{\partial f_{ir}(\boldsymbol{y}_{i,-r}, \boldsymbol{x}_i, \boldsymbol{L}_i)}{\partial y_{ig}}\right)$$

Next, we introduce the following lemma.

Lemma 2. *Total derivative of* $mc_{ig} = mc_{ir}\left(-\frac{\partial f_{ir}(\boldsymbol{y}_{i,-r}, \boldsymbol{x}_i, \boldsymbol{L}_i)}{\partial y_{ig}}\right)$

$$d\log p_{ir}/p_{ig} = d\log \mu_{ir}/\mu_{ig} + d\log y_{ig} - d\log y_{ir} + \sum_k \left(-\frac{\partial \log f_{ir}\left(\boldsymbol{y}_{i,-r}, x\right)}{\partial \log y_{ig} \partial \log y_k}\right) d\log y_k + \\ + \sum_{j,p}\left(-\frac{\partial \log f_i\left(\boldsymbol{y}_{-g}, \mathbf{x}_i\right)}{\partial \log y_{ig}\partial \log x_{i,jp}}\right) d\log x_{i,jp} + \sum_f \left(-\frac{\partial \log f_i\left(\boldsymbol{y}_{-g}, \mathbf{x}_i\right)}{\partial \log y_{ig}\partial \log L_{if}}\right) d\log L_{if} \tag{6}$$

Proof. first-order approximation of $mc_{ig} = mc_{ir}\left(-\frac{\partial f_{ir}\left(\mathbf{y}_{i,-r}, \mathbf{x}_i, \mathbf{L}_i\right)}{\partial y_{ig}}\right)$

$$\begin{aligned} d\log mc_{ig} =& d\log mc_{ig} + d\log y_{ig} - d\log y_{ir} + d\log\left\{-\frac{\partial \log f_{ir}\left(y_{i-r}, x\right)}{\partial \log y_{ig}}\right\} \\ =& d\log mc_{ig} + d\log y_{ig} - d\log y_{ir} + \sum_k\left(-\frac{\partial \log f_{ir}\left(y_{i,-r}, x\right)}{\partial \log y_{ig}\partial \log y_k}\right) d\log y_k \\ &+ \sum_{j,p}\left(-\frac{\partial \log f_i\left(y_{-g}, \mathbf{x}_i\right)}{\partial \log y_{ig}\partial \log x_{i,jp}}\right) d\log x_{i,jp} + \sum_f\left(-\frac{\partial \log f_i\left(y_{-g}, \mathbf{x}_i\right)}{\partial \log y_{ig}\partial \log L_{if}}\right) d\log L_{if} \end{aligned}$$

□

This formula is valid for all classes of joint production. In general, we find that determining prices requires cross-elasticity between production and inputs. For later proof, we define the LHS of the equation 6 as Θ_{ig}.

$$d\log p_{ig}/p_{ir} = \Theta_{ig} \tag{7}$$

Then, we proceed to the main proof.

Proof. From Lemma 1 We know for one reference product r:

$$\frac{q_{ir} mc_{ir}}{C_i\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)} d\log mc_{ir} = -d\log A_i + \sum_{j,p}\frac{p_{jp}x_{jp}}{C_i\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)} d\log p_{jp} + \sum_f \frac{w_f L_{if}}{C_i\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)} d\log w_f + \sum_{g\neq r}\left(-\frac{q_{ig} mc_{ig}}{C_i\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)}\right) d\log mc_{ig} \iff \\ \frac{q_{ir} mc_{ir}}{C_i\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)} d\log p_{ir} = -d\log A_i + d\log \mu_i + \sum_{j,p}\frac{p_{jp}x_{i,jp}}{C_i\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)} d\log p_{jp} + \sum_f \frac{w_f L_{if}}{C_i\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)} d\log w_f \\ + \sum_{g\neq r}\left(-\frac{q_{ig} mc_{ig}}{C_i\left(\boldsymbol{q}_i, \boldsymbol{p}_i\right)}\right) d\log p_{ig} \tag{8}$$

From equation 7, we have

$$d\log p_{ig}/p_{ir} = \Theta_{ig}$$

Combining 5 with 7 yields

$$\frac{q_{ir}mc_{ir}}{C_i\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}d\log p_{ir} = -d\log A_i + \sum_g \frac{q_{ig}mc_{ig}}{C_i\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}d\log\mu_{ig} + \sum_{j,p}\frac{p_{jp}x_{jp}}{C_i\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}d\log p_{jp} + \sum_f \frac{w_f l_{if}}{C_i\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}d\log w_f$$
$$+\sum\left(-\frac{q_{ig}mc_{ig}}{C_i\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}\right)d\log p_{ig} \iff$$
$$\frac{q_{ir}mc_{ir}}{C_i\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}d\log p_{ir} = -d\log A_i + d\log\mu_i + \sum_{j,p}\tilde{\Omega}_{ig,jp}d\log p_{jp} + \sum_f \tilde{\Omega}_{ig,f}d\log w_f + \sum_{g\neq r}\left(-\frac{q_{ig}mc_{ig}}{C_i\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}\right)\left[d\log p_{ir} + \Theta_{ig}\right] \iff$$
$$d\log p_{ir} = -d\log A_i + d\log\mu_i + \sum_{j,p}\tilde{\Omega}_{ig,jp}d\log p_{jp} + \sum_f \tilde{\Omega}_{ig,f}d\log w_f + \sum_{g\neq r}\left(-\frac{q_{ig}mc_{ig}}{C_i\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}\right)\Theta_{ig}$$

Since $\Theta_{ig} = 0$ if g is an reference products, the price equations could be written by

$$d\log p_{ig} = -d\log A_i + d\log\mu_i + \sum_{j,g'}\tilde{\Omega}_{ig,jg'}d\log p_{jg'} + \sum_f \tilde{\Omega}_{ig,f}d\log w_f + \left\{\mathbb{I}_i(g) - \sum_{g\neq r}\left(\frac{q_{ig}mc_{ig}}{C_i\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}\right)\right\}\Theta_{ig}$$

In vector notation

$$d\log\boldsymbol{p} = (I - \tilde{\Omega}^{\mathcal{NG}\times\mathcal{NG}})^{-1}\left\{-d\log\boldsymbol{A}^{\mathcal{NG}\times 1} + d\log\boldsymbol{\mu}^{\mathcal{NG}\times 1} + \tilde{\Omega}_f^{\mathcal{NG}\times\mathcal{F}}d\log\boldsymbol{w} + (\mathbf{1}-\mathbf{C})\circ\boldsymbol{\Theta}^{\mathcal{NG}\times 1}\right\},$$

where $\circ$ represents Hadamard product and and C is a vector of $\mathcal{NG}\times 1$, with the following C_i common elements for firm $i \in \mathcal{N}$,

$$C_i = \sum_{g\neq r}\left(\frac{q_{ig}mc_{ig}}{C\left(\boldsymbol{q}_i,\boldsymbol{p}_i\right)}\right)$$

We know

$$d\log Y = -\boldsymbol{b}'d\log\boldsymbol{p}$$

$$d\log Y = -\boldsymbol{b}'\tilde{\Psi}^{\mathcal{NG}\times\mathcal{NG}}\left\{-d\log\boldsymbol{A} + d\log\boldsymbol{\mu} + \tilde{\Omega}_f d\log\boldsymbol{w} + (\mathbf{1}-\mathbf{C})\circ\boldsymbol{\Theta}^{\mathcal{NG}\times 1}\right\} \iff$$
$$= -\tilde{\boldsymbol{\lambda}}'\left\{-d\log\boldsymbol{A} + d\log\boldsymbol{\mu} + \tilde{\Omega}_f d\log\boldsymbol{w} + (\mathbf{1}-\mathbf{C})\circ\boldsymbol{\Theta}^{\mathcal{NG}\times 1}\right\}$$

subtracting $\sum_f \tilde{\Lambda}_f d\log L_f$ from both sides yields

$$d \log TFP = \sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log A_i - \sum_{i \in \mathcal{N}} \tilde{\lambda}_i d \log \mu_i - \sum_{f \in \mathcal{F}} \tilde{\Lambda}_f d \log \Lambda_f$$
$$- \sum_i \left(\sum_{g \in \mathcal{G}} \tilde{\lambda}_{ig} d \log p_{ig}/p_{ir} - \sum_{g \neq r} \frac{q_{ig} mc_{ig}}{C_i (\boldsymbol{q}_i, \boldsymbol{p}_i)} \tilde{\lambda}_i d \log p_{ig}/p_{ir} \right)$$

$$\begin{aligned}
\left(\sum_g \tilde{\lambda}_{ig} d \log p_{ig}/p_{ir} - \sum_{g \neq r} \frac{q_{ig} mc_{ig}}{C_i (y, p)} \tilde{\lambda}_i d \log p_{ig}/p_{ir} \right) &= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} s_{ig} d \log p_{ig}/p_{ir} - \sum_{g \neq r} c_{ig} d \log p_{ig}/p_{ir} \right) \\
&= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} s_{ig} d \log p_{ig}/p_{ir} - \sum_{g \in \mathcal{G}} c_{ig} d \log p_{ig}/p_{ir} \right) \\
&= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} \left(s_{ig} - c_{ig} \right) d \log p_{ig} \right) \\
&= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} \left(s_{ig} - \frac{\frac{q_{ig} mc_{ig}}{C(y,p)}}{s_{ig}} s_{ig} \right) d \log p_{ig} \right) \\
&= \tilde{\lambda}_i \left(\sum_{g \in \mathcal{G}} \left(s_{ig} - \frac{1}{\gamma_{ig}} s_{ig} \right) d \log p_{ig} \right) \\
&= \tilde{\lambda}_i \left(E_{\boldsymbol{s}_i} \left[d \log \boldsymbol{p}_i \right] E_{\boldsymbol{s}_i} \left[\frac{1}{\boldsymbol{\gamma}_i} \right] - E_{\boldsymbol{s}_i} \left[d \log \boldsymbol{p}_i, \frac{1}{\boldsymbol{\gamma}_i} \right] \right) \\
&= -\tilde{\lambda}_i Cov_{\boldsymbol{s}_i} \left(d \log \boldsymbol{p}_i, \frac{1}{\boldsymbol{\gamma}_i} \right)
\end{aligned}$$

where $c_{ig} = \frac{q_{ig} mc_{ig}}{C(\boldsymbol{q}_i, \boldsymbol{p}_i)}$.

□

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