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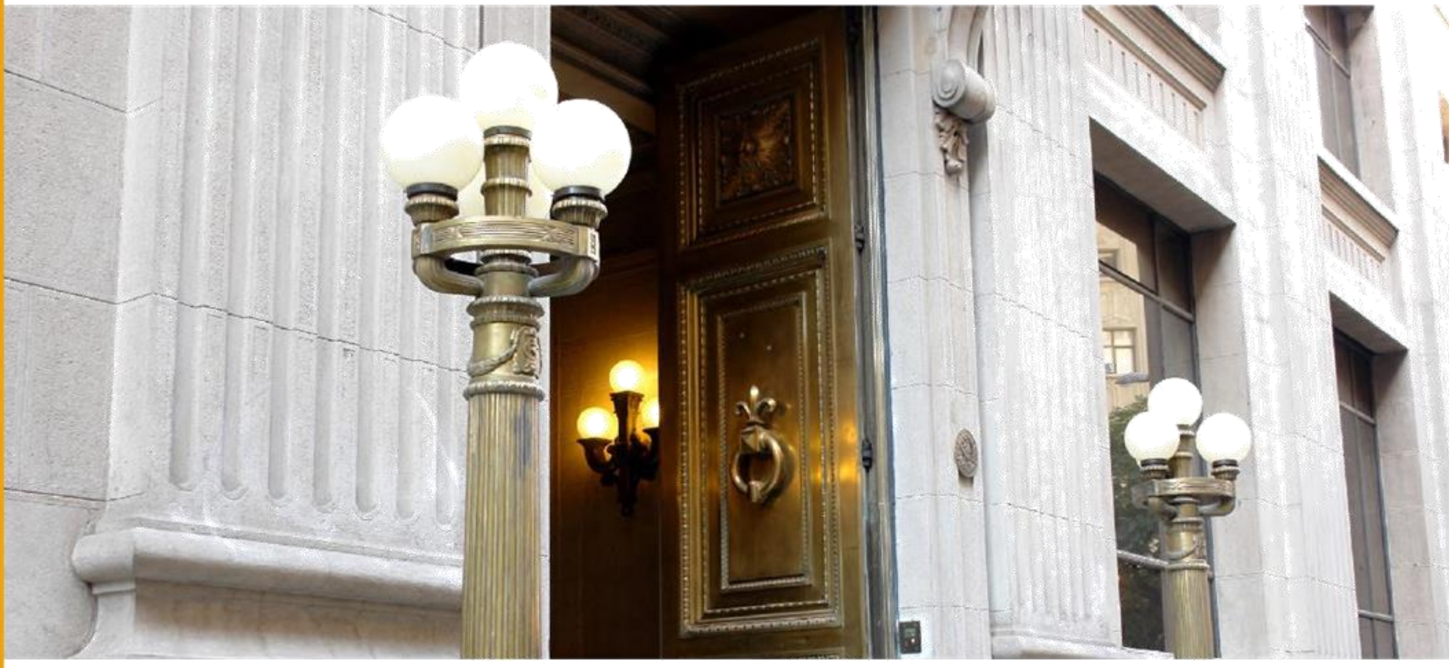

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Tail-Risk Indicators with Time-Variant Volatility Models: the case of the Chilean Peso*

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Resumen

En este artículo, proponemos un marco para construir indicadores de riesgo de cola para el peso chileno (CLP) basados en modelos de volatilidad variantes en el tiempo [p. ej., Engle (1982), Taylor (1982), Nelson (1991), Heston y Nandi (2000)], que estimamos combinando: (i) rentabilidades diarias, (ii) volatilidad implícita de opciones (VI) y (iii) volatilidad realizada intradía (VR). Nuestros resultados empíricos muestran que el ajuste de los modelos en la muestra mejora al incorporar medidas de volatilidad (VI o VR). Proporcionamos una aplicación del marco para evaluar escenarios extremos.

Abstract

In this paper we propose a framework for building tail-risk indicators for the Chilean Peso (CLP) based on time-variant volatility models [e.g., Engle (1982), Taylor (1982), Nelson (1991), Heston and Nandi (2000)], which we estimate by combining: (i) daily returns, (ii) option-implied volatility (IV), and (iii) intraday realized volatility (RV). Our empirical results show that the in-sample fit of the models improves when volatility measures (IV or RV) are added. We provide an application of the framework to evaluate extreme scenarios.

*The views expressed herein are those of the authors and should not be attributed to the IMF, its Executive Board, or IMF management, neither those of the Central Bank of Chile or its board members.

I. Introduction

The foreign exchange (FX) spot market represents great importance at the global level due to its depth and its fundamental role in all types of economies, playing a key role in international trade by allowing the conversion of one currency to another. Moreover, the FX spot market is the largest financial market in the world, trading around US$7,500 billion, and rapidly growing in the last decade and actively trading in financial hubs such as New York, London, and Hong Kong. For the proper assessment of this market functionality and tail-risks, time-variant volatility is an important element to take into consideration. A standard approach to deal with this is to consider a GARCH model, quite popular in finance since it allows to have conditional volatility as a function of squared past returns. Hansen and Lunde (2005) found that a simplest GARCH(1,1) performs relatively well in term of out-of-sample forecast for a representative exchange rate, being hard to beat against other more complex GARCH models such as: (i) Nonlinear GARCH (NGARCH), proposed by Gloster et al. (1993), (ii) Threshold GARCH (TGARCH), proposed by Zakoian (1994), or (iii) Exponential GARCH (EGARCH) proposed by Nelson (1991); however, GARCH models with leverage effect seems to be more appropriate for stock returns (Engle et al., 1987). In contrast to GARCH model, Stochastic volatility (SV) models include an additional source of randomness that affects the second conditional moment of the returns; therefore, volatility is not fully explained by the past returns. Heston (1993) proposes an SV model in continuous time which models the variance as stationary process that is affected by a shock that is correlated with the returns-shock. It was rapidly adopted by practitioners due to ability to offer a closed-form expression for pricing European-type options (Gatheral, 2006). There are two SV models in discrete time: (i) Heston and Nandi (2000) which is based on Heston's model but imposes perfect correlation between the errors of the returns and volatility equations as in the case of GARCH models, and (ii) Taylor (1982) which models the logarithm of the variance as an autoregressive process with a shock that is uncorrelated with the returns-shock; the model can be approximated and estimated with a linear Gaussian state-space.

In the case of the Chilean Peso (CLP), Ceballos (2010) and Gonzales and Oda (2015) estimate the Risk-Neutral-Density functions implied in the option-based contracts on the CLP given that these contracts contain relevant information about the underlying asset and the probability of risky events, making them an interesting tool at the time of analyzing the market's belief. Following Malz (1997), they use three types of option-based contracts: at the money (ATM), risk reversal, and strangle

(Butterfly). The latter also reports how these contracts can be used to predict significant movements of the CLP in the following 3 months. An empirical approach is presented in Jara and Piña (2023), they use a two-state Markov Switching EGARCH model with t-Student errors to fit the logarithm returns on CLP for the period January 2000 – February 2020. From their results: (i) the estimate of degree of freedom shows that tail-events are relevant, and (ii) the logarithm of the conditional variance is very persistent, having autocorrelations of 97% and 90% in the low and high-volatility states, respectively. In addition to the option-based contracts and returns, intraday measures can be used to estimate a density function for the CLP.

This paper has three contributions. First, we introduce the daily realized volatility (RV), providing statistical indicators that can be used to assess if the market is functioning properly. Our measure of RV is the standard deviation scaled by 1000. Based on a sample period from January 4th, 2010 — July 15thm 2025, the 90% confidence interval for liquid hours (09:00-13:00) is [0.07; 0.25]. Second, we estimate the parameters of several time-variant volatility models for the CLP: (i) standard GARCHs (GARCH, NGARCH, and TGARCH), (ii) the EGARCH, (iii) the Heston-Nandi (approximated by an Asymmetric GARCH), and (iv) the Taylor models (also known as Stochastic Volatility model). We combine 3 sets of information on the estimation: (i) daily returns, (ii) implied volatility (IV) obtained from ATM option contracts, and (iii) daily RV. Our results show, in line with Li et al. (2024), that both IV and RV contain useful information to improve in-sample fitting of the models and therefore volatility estimates. Third, based on these findings, and following Lafarguette and Veyrune (2021) we propose a framework to quantify tail-risk events using Monte Carlo (MC) simulations, and we provide examples on how to apply it in the context of evaluating extreme scenarios.

The paper is organized as follows. Section II provides an overview of the model and the techniques to estimate it. Section III describes the intraday measures of RV and the IV measures. Section IV discusses empirical results. Section V illustrates the proposed framework for tail-risk indicators. And Section VI concludes.

II. Models

In this section we first discuss GARCH, NGARCH and TGARCH models, and the Heston-Nandi model, all of them model the conditional variance. Then we briefly introduce the EGARCH model which assumes a structure for the logarithm of the conditional variance. We finish the section with the Stochastic Volatility (SV) model.

1. GARCH models

We consider the Nonlinear GARCH (NGARCH) and the Threshold GARCH (TGARCH) along with the standard GARCH model, those can be written as follows:

$$r_t = \psi + \varepsilon_t \tag{1}$$

$$h_t = \omega + \alpha(\varepsilon_{t-1} - \tau)^2 + \beta h_{t-1} + \gamma \varepsilon_{t-1}^2 d_{t-1} + \delta x_{t-1} \tag{2}$$

where r_t is the daily return of CLP, h_t is the conditional variance, $\varepsilon_t \sim N(0, h_t)$ is the error term, d_t is a dummy variable that takes the value of one when ε_t is positive and zero otherwise, and x_t is an exogenous variable with long-run average of $\bar{x}$. We consider as exogenous variables the ex-post realized volatility (RV) and ex-ante implied volatility (IV). As we discussed in Section III, the RV is obtained from all the daily transactions executed through Datatec, which is a platform used mainly by local banks, and the IV is obtained from Bloomberg as the quote for the at-the-money contracts over the CLP to a given horizon.

The parameters of the model are ψ, ω, α, β, γ, and δ. The persistence of the conditional variance is $\lambda = \alpha + \beta + 0.5\gamma$, meanwhile the long-run (LR) volatility is $\sigma_{LR} = \sqrt{(\omega + \alpha\tau^2 + \delta\bar{x})/(1-\lambda)}$. Given that the restrictions are $\omega > 0$, $\alpha > 0$, β>0, $0 < \lambda < 1$, and $\omega + \alpha\tau^2 + \delta\bar{x} > 0$. Finally, this GARCH model collapses to: (i) NGARCH when $\gamma = 0$, (ii) TGARCH when $\tau = 0$, and (iii) GARCH when $\tau = \gamma = 0$ (StataCorp, 2023).

Heston and Nandi (2000) proposed a discrete time version of Heston's (1993) model, which is very popular among financial analysts (Gatheral, 2006). It belongs to GARCH model family and for that it imposes perfect correlation between shocks (associated to the return and the volatility equations). The proposed model is described by the following equations:

$$r_t = \psi + \sqrt{h_t} z_t \tag{3}$$

$$h_t = \omega + \alpha\left(z_{t-1} - \gamma\sqrt{h_{t-1}}\right)^2 + \beta h_{t-1} + \delta x_{t-1} \tag{4}$$

where h_t is the conditional variance (as before), and $z_t \sim N(0, 1)$ is the error term. Given this and the structure of model, the Moment Generating Function (MGF) can be obtained recursively. It should be noted that (4) is not equivalent to the (2) as in Heston-Nandi the standardized shock is used instead of ε_t. This feature allows to have a closed-form MGF. Moreover, Mazzoni (2010) suggests an approximation of the recursion when α is small, this allows us to obtain the MGF in closed form, and therefore to compute the skewness and kurtosis in one-step. The author proposes to use these moments in a Gram-Charlier density (see Appendix A).

An alternative specification of (1)-(2) is presented in Christoffersen et al. (2014) in which an additional equation is added for the RV, fitting an AR(1) process for the squared measure, and the Heston-Nandi model is used for the returns. In such case there is an extra error term that is correlated with the one presented in the return equation (ε_t). Their empirical results, based on the S&P 500 futures prices for the period January 2nd, 1990 — December 30th, 2010, show that such correlation is small (around 10%); meanwhile the weights for h_t obtained from returns and h_t obtained from RV information are 40% and 60%, respectively. In our case, the assessment of the relevance of IV/RV information will be obtained from the statistical fit of the model.

Expanding (4): $h_t = (\omega + \alpha) + \alpha(z_{t-1}^2 - 1) - 2\alpha\gamma z_{t-1}\sqrt{h_{t-1}} + (\alpha\gamma^2 + \beta)h_{t-1} + \delta x_{t-1}$, thus the second and the third elements have zero unconditional expected value. We note that the persistence is $\lambda = \alpha\gamma^2 + \beta$, and the LR volatility is $\sigma_{LR} = \sqrt{(\omega + \alpha + \delta\bar{x})/(1 - \lambda)}$. In contrast to previous GARCH models, the parameter ω in Heston-Nandi model could be zero. Indeed, empirical results presented in several papers (Mazzoni, 2010; Christoffersen et al., 2014; Escobar-Anel et al., 2022; Mozumder et al., 2024) show that estimates for ω and α are very small, the latter around 4e-6 or 5e-8 and the former between 3e-9 and 3e-19; meanwhile γ tend to be very large, between 100 and 720.[1] Based on these findings we make following two assumptions in the current application: (i) $\omega + \alpha \approx \alpha$, and (ii) $\alpha(z_{t-1}^2 - 1) \approx 0$. These are valid under the case that the parameter α is small but positive.

Given previous discussion, an empirical counterpart of (3)-(4) is the Asymmetric GARCH (AGARCH), which is defined by the following equations:

$$r_t = \psi + \varepsilon_t \tag{5}$$

$$h_t = \omega_* + \alpha_* \varepsilon_{t-1} + \beta_* h_{t-1} + \delta x_{t-1} \tag{6}$$

where $\varepsilon_t \sim N(0, h_t)$ is the error term (as before), $\omega_* = \alpha$, $\alpha_* = -2\alpha\gamma$, and $\beta_* = \alpha\gamma^2 + \beta$.

Finally, the EGARCH model (Nelson, 1991) is the natural alternative for the SV model, given that explicitly expresses the logarithm of the variance:

$$r_t = \psi + \sqrt{h_t} z_t \tag{7}$$

$$\log(h_t) = \kappa_0 + \kappa_1 \log(h_{t-1}) + \kappa_2 z_{t-1} + \kappa_3 \tilde{z}_{t-1} + \kappa_x \log(x_{t-1}) \tag{8}$$

where h_t is conditional variance, $z_t \sim N(0,1)$, and $\tilde{z}_{t-1} \equiv |z_{t-1}| - \sqrt{2/\pi}$.

The EGARCH has been used as an auxiliary model to estimate the SV model, which we review in the next subsection, using indirect inference. Thus, having empirical results allow us to use the model as a bridge between GARCH family and SV model. Interesting, Jara and Piña (2023) fit a two-state EGARCH model for the CLP and Lafarguette and Veyrune (2021) fit a EGARCH model for the Mexican Peso (MXN); in both empirical applications t-Student errors are used, accommodating extreme events.

[1] Mazzoni (2010) estimates the model using S&P 500, Dow Jones, Hang Seng, and DAX returns from September 2004 to August 2008. Christoffersen et al. (2014) estimate the parameters of the model using S&P 500 daily return from January 1990 to December 2010. Escobar-Anel et al. (2022) use the sets of parameters, estimated by other authors which are based on S&P 500 daily returns from January 1992 to December 1994, and from January 1989 to December 2001. Mozumder et al. (2024) use S&P 500 weekly return from January 2016 to December 2018.

2. SV model

Following Kim et al. (1998), the underlying process can be described by the following system of equations:[2]

$$r_t = \exp(\mu/2 + m_t/2)\, u_t \tag{9}$$

$$m_t = \phi m_{t-1} + \delta x_{t-1} + \eta v_t \tag{10}$$

where u_t and v_t are uncorrelated standard normal shocks, x_t is an exogenous variable, and m_t is the logarithm of the variance. Also, we assume a stationary process for the log-variance, meaning that the condition of $|\phi| < 1$ is satisfied.

The SV model was introduced in Taylor (1982) as an alternative to ARCH models for modeling financial series and its statistical properties are discussed in Taylor (2008). Hull and White (1987) use a similar model, in continuous-time, to price options (imposing zero-correlation between return and volatility shocks), but it has been replaced by the model proposed by Heston (1993) which provides a closed-form solution for pricing European options. Given this feature, there is an extensive literature regarding the estimation of these models based on returns, and option prices finding that combining the two source of information improves the estimation of the key parameters of the models.[3] Alternatively, other studies use intraday information to achieve this goal (Byun et al. 2020, Li et al. 2024). In our empirical application will follow that approach.

The model (9)-(10) is a non-linear state-space which can be approximated as linear Gaussian by considering the variable $y_t = \log(r_t^2 + \Delta)$, where Δ is a small constant[4], included to avoid numerical problems when returns are close to zero. This implies $y_t \cong \mu + m_t + \log(u_t^2)$, the last term could be approximated by a normal distribution or by a mixture of normal disturbances.

[2] We impose zero mean for the daily returns. Usually (10) is named as volatility equation. For a comprehensive review of the model see Broto and Ruiz (2004).

[3] See for example: Christoffersen et al. (2013), Hao and Zhang (2013), Kannianen et al. (2014), Zhang and Zhang (2020), Escobar-Anel et al. (2022), and Alfaro and Inzunza (2023). In the latter, the estimation of the parameter associated to the risk-premium of investing in equity vis-à-vis in short-term interest rate increases when option-based indexes are included.

[4] In the empirical application we set $\Delta = 0.0001$ in line with Chan and Hsiao (2014), but also according with Fuller's transformation, which is explained in detail in Appendix A.

In the former we have $y_t = (\mu - 1.2704) + m_t + \sqrt{\pi^2/2}\, e_t$, where $e_t \sim N(0,1)$ and $\pi^2/2 = 4.935$ is the variance.[5] Kim et al. (1998) emphasized that QMLE with the normal approximation is consistent and asymptotically normal distributed, according with the use of the Kalman Filter (KF), but it performs poorly in finite-samples. In their application with 946 observations, they implemented a 7-component mixture of normal disturbances to get a good approximation of the distribution of $\log(u_t^2)$.

Alternatively, the mixture is calibrated before to estimate the parameter of the SV model, which is an advantage, and the estimation is obtained by means of Monte Carlo Markov Chain (MCMC). Abbara and Zevallos (2019) suggest an alternative approximation using mixtures of 2 and 3 normal disturbances but implying that means and variances should be estimated along with the parameters of the SV model. In this case QMLE is performed by means of a modified KF. Their Monte Carlo results show that the method provides small biases on the estimates of the parameters of the SV model, including cases when u_t is a heavy-tail distribution, such as a t-Student. Abbara and Zevallos (2023) extend the previous analysis for the cases of asymmetric distributions.

Given that, in this study: (i) we use a linear space-state consistently with the large sample available for the estimation of the model, and (ii) we provide additional results when the variance of e_t is different than one, allowing fat tail distributions.

As it was discussed above, many of the empirical applications of the SV model involves the use of Bayesian methods (Kim et al., 1998; Broto and Ruiz, 2004; Chan and Hsiao, 2014). Uninformative priors lead to the following posteriors:[6] (i) inverse-gamma distribution for η, and (ii) normal distribution for μ and ϕ; starting from an initial series for m_t which is the logarithm of the sample variance of the returns.

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[5] If u_t is standard normal, then the mean and the variance of $\log(u_t^2)$ are -1.2704 and 4.935, respectively; however, if u_t is distributed t-Student the mean decreases and the variance increase as the degrees of freedom (dof) decreases. We explore the latter by using 10'000 Monte Carlo simulations with samples of 50'000 observations, we found that the mean and the variance of $\log(u_t^2)$ are: (i) -1.17 and 5.16 for 10 dof, (ii) -1.06 and 5.43 for 5 dof, and (iii) -0.82 and 6.13 for 2.5 dof.

[6] See Appendix A for an intuition about these posteriors.

III. Volatility Measures

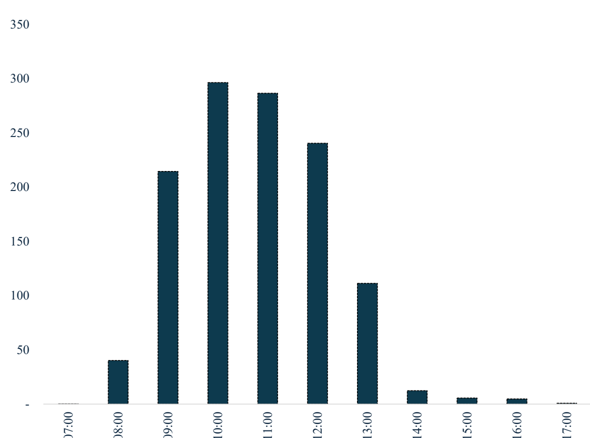
In this section we discuss the intraday volatility measure, and the patterns observed for this variable in high frequency. This will help uncover volatility dynamics that exist in the Chilean FX market, to further characterize this market and give a better understanding of it. We also discuss the option-implied volatilities.

1. Realized Volatility

Figure 1 shows the daily average traded amount on Datatec from January 4, 2010 – July 15, 2024.[7] It is noteworthy that there are times of greater market depth, and where most of the operations occur. We might expect that periods of low liquidity by nature will be more volatile, it is considered to use those operations that occur in liquid hours between 09:00 and 13:00. Nevertheless, we consider an extended period 08:00 to 16:00 to compute the daily realized volatility (RV), which is the standard deviation of returns of CLP within a given time-interval, scaled by 1,000.

Figure 1: Transactions on Datatec

(USD million, daily averages)

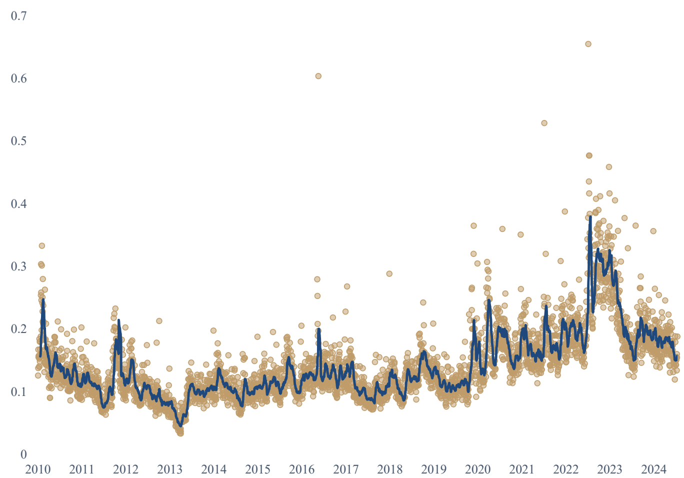


[7] Datatec is an electronic transactional system for the foreign exchange market and its derivatives, through which banks, stockbrokers and securities agents in Chile are interconnected. It allows trading in the interbank and non-bank dollar market. Since banks are the main intermediaries in the local market, the live price presented in this platform is considered the "Live" price. Other market participants opt to use the platform as an observation window of such price. On average formal FX market operators trade through Datatec around US$ 1.3 billion. This platform is also currently used by the Ministry of Finance and the Central Bank of Chile for their respective FX auctions.

Figure 2 shows the RV, where we can appreciate that volatility has increased in the last 5 years, especially during the pandemic period (2020-2022) which can be considered a turbulent period, it is noticeable more volatile in every window of time, which can be related to the poor price formation during the period, and more ample bid-ask spreads.

Figure 2: Realized Volatility

(index, moving average 5 business days)



As mentioned before, RV is based on intraday transactions and there is different degree of market liquidity during the day. Table 1 shows the percentiles by hours during the sample period (January 4, 2010 – July 15, 2024). For example, the median RV observed between 9:00 and 10:00 is 0.13, which is similar to the median observed around midday (between 12:00 and 13:00) 0.13. Based on that, we confirm that during the liquid hours (09:00 to 13:00) the RV fluctuates, most of the time, between 0.08 and 0.20 (10 and 90 percentiles, respectively). Other percentiles can be used to build narrow ranges (e.g., 75 minus 25) to assess if a current level of RV is in line with the historical performance.

Figure 3 shows the RV in hourly averages during different sample periods. We note that periods 2010-2012, 2013-2016, and 2017-2019 have similar averages during liquid hours (09:00-13:00); however, after 2019 these averages increase which might be related to the higher uncertainty levels in the country.

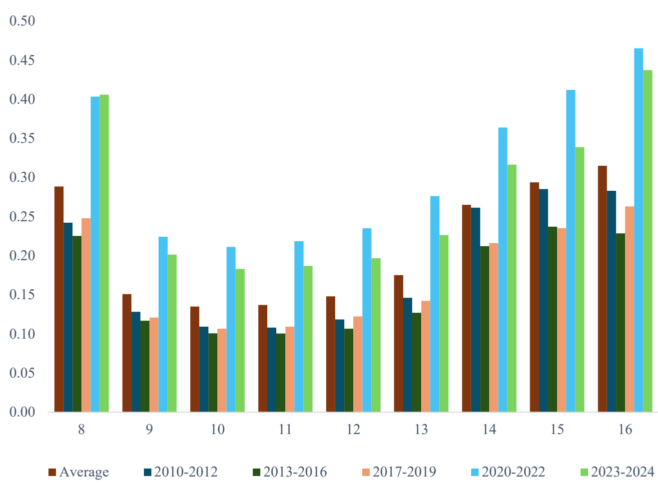
Table 1: Intraday Realized Volatility

(index)

	8:00	9:00	10:00	11:00	12:00	13:00	14:00	15:00	16:00
95	0.571	0.251	0.236	0.239	0.262	0.322	0.519	0.621	0.728
90	0.460	0.216	0.196	0.203	0.219	0.264	0.428	0.504	0.552
75	0.334	0.169	0.152	0.154	0.167	0.195	0.311	0.351	0.366
50	0.248	0.131	0.115	0.115	0.126	0.145	0.226	0.242	0.240
25	0.185	0.105	0.092	0.092	0.098	0.113	0.164	0.161	0.159
10	0.147	0.089	0.077	0.077	0.080	0.091	0.117	0.110	0.107
5	0.128	0.079	0.069	0.067	0.070	0.080	0.092	0.075	0.083
90-10	0.313	0.127	0.119	0.126	0.138	0.173	0.311	0.395	0.445
75-25	0.149	0.064	0.060	0.062	0.069	0.082	0.146	0.189	0.207

Figure 3: Realized Volatilities

(index, hourly averages)



2. Implied Volatility

The use of IV's for estimating the density function of FX was proposed by Malz (1997). The author simplifies the analysis of the volatility smile by considering 3 option-combination contracts which were popular among dealers: at the money (ATM) or straddle, risk reversal (RR), and strangle or butterfly. The first one is a contract that includes a call and a put at the same exercise price, usually at the forward, and it offers a full coverage over any discrepancy of the future FX. As it is expected an increase in the volatility of the FX implies a higher cost of the contract, therefore the ATM is the first order element in any volatility smile. Secondly, RR is a strategy that results from combining a long position in a call option with an exercise price greater than the forward and a short position in a put option with an exercise price lower than the forward, both with the same delta. Finally, strangle contract is a combination of purchasing a call with a strike price greater than the forward and a put with a strike price lower than the forward, both with the same term and the same delta. The latter implies that the exercise price of both options has the same distance from the forward price.

Gonzales and Oda (2015) estimate the density function of CLP by considering the 3 contracts (ATM, RR and strangle). That density is known as risk-neutral given that is obtained from option prices instead of actual prices of the underlying asset. It is important to stress that the probabilities obtained from the risk-neutral density are different from the actual probabilities. Abdymomunov et al. (2023) propose a parametric approach to combine both set of probabilities with an application to assess the plausibility of a stressed scenario. In this study we focus our analysis on the empirical density, and therefore the use of the IV measures is justified only by statistical fitting.

Figure 4 shows the ATM-IV for 1, 3, and 6 months during the period January 2010 – August 2024. The averages for ATM-IV are 12.22%, 12.25%, and 12.30%, respectively. The medium-term measure (6M: 6 months) is smoother than the short-term one (1M: 1 month); however, the difference in levels reflects uncertainty about interest rate differential between Chile and US as well as volatility of the CLP. Several events are highlighted with sudden increases in the IV measures, including the European crisis in 2012, the local social unrest in 2019, the Covid crisis in 2020, the local pension funds withdrawals during 2020-2021, and the local plebiscite for a new constitution in 2022.

Finally, Table 2 provides descriptive statistics that will be useful for the next section where RV & IV, and their logarithms, are exogenous variables.

Figure 4: ATM Implied Volatilities

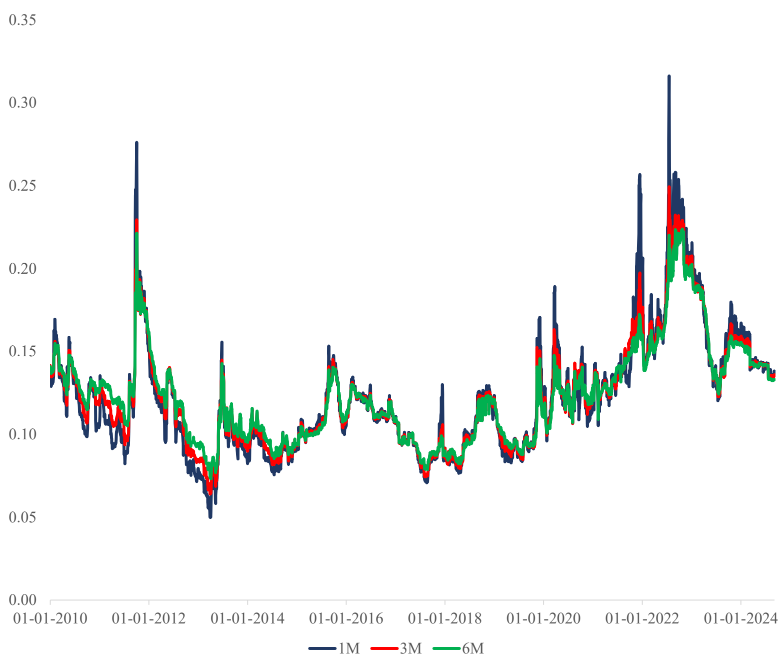


Table 2: Descriptive IV & RV

	RV	log RV	IV			log IV		
			(1m)	(3m)	(6m)	(1m)	(3m)	(6m)
Mean	0.14163	-2.02567	0.12214	0.12240	0.12297	-2.14164	-2.13089	-2.12147
Std Dev	0.05960	0.36735	0.03600	0.03144	0.02896	0.27461	0.24314	0.22322
Min	0.03332	-3.40149	0.04990	0.06430	0.07310	-2.99773	-2.74420	-2.61593
Max	0.98948	-0.01057	0.31620	0.24940	0.22390	-1.15138	-1.38870	-1.49656

IV. Empirical Results

In this section we discuss the results obtained for both models: GARCH and SV using daily log-return of CLP, and the volatility measures described in the previous section. Our sample runs from Jan 4, 2010, to July 15, 2024, having 3622 data points for all estimates.

1. GARCH Models

Table 3 shows the result for standard GARCH with exogenous variables (RV and IV) in the variance equation. The first column shows that the model without exogenous, the second with RV, 3rd to 5th with IV (different tenors), and 6th to 8th with both volatility measures.[8] The overall in-sample fits, measured by BIC, indicates that either RV or IV should be included; however, the model with RV and IV 1-month provides a better fit.[9] Tables 4 and 5 show the results for extension of the standard GARCH model (NGARCH and TGARCH). The additional parameter is significant in all the specifications evaluated; however, the overall in-sample is similar for the case when both RV and IV 1-month are included.

As we discussed in Section II, the Heston-Nandi model is widely used in Finance for pricing option as it offers a convenience structure for the conditional variance. Table 6 shows the results for AGARCH models, which is an approximated version of the Heston-Nandi. The results confirm that either RV or IV improve the in-sample fit of model; as before, combining RV with IV 1-month provides the better fit. However, the results of the AGARCH model are not consistent with the Heston-Nandi given that $\alpha < 0$.

[8] Alternatively, variance measures (IV and RV squared) could be considered instead.

[9] Following Jara and Piña (2023) we consider splitting the sample according with two-state regimen. Results for GARCH models, without exogenous variables, are reported in Appendix B using sub-samples.

Table 3: QMLE for GARCH models

			IV			RV-IV		
		RV	(1m)	(3m)	(6m)	(1m)	(3m)	(6m)
ψ	0.016251	0.012367	0.021489	0.015172	0.014364	0.019048	0.0179	0.01739
	(0.0096)	(0.0101)	(0.0101)	(0.0098)	(0.0098)	(0.0101)	(0.0101)	(0.0101)
α	0.055425	0.115133	0.080282	0.092445	0.089142	0.082368	0.099425	0.105803
	(0.0039)	(0.0130)	(0.0162)	(0.0107)	(0.0097)	(0.0147)	(0.0135)	(0.0134)
β	0.936308	0.294449	-0.22937	0.736973	0.793513	-0.01958	0.120371	0.150929
	(0.0053)	(0.0527)	(0.1181)	(0.0348)	(0.0262)	(0.0597)	(0.0731)	(0.0666)
ω	0.004431	-2.70266	-2.67095	-4.62566	-5.07753	-2.82933	-3.06007	-3.11589
	(0.0010)	(0.1096)	(0.1345)	(0.2031)	(0.2279)	(0.1189)	(0.1444)	(0.1466)
δ_{IV}			15.59167	16.22609	16.90617	10.02967	9.431044	8.759396
			(0.7508)	(1.1131)	(1.3027)	(1.1745)	(1.2774)	(1.2687)
δ_{RV}		8.984034				4.410055	5.281629	5.899942
		(0.4634)				(0.6586)	(0.6478)	(0.6137)
LL	-3523	-3497	-3483	-3493	-3498	-3464	-3472	-3477
AIC	7055	7004	6976	6996	7007	6940	6957	6967
BIC	7080	7035	7007	7027	7038	6978	6994	7004

Table 4: QMLE for NGARCH models

			IV			RV-IV		
		RV	(1m)	(3m)	(6m)	(1m)	(3m)	(6m)
ψ	0.021908	0.011889	0.018223	0.018529	0.018983	0.016529	0.015924	0.015712
	(0.0101)	(0.0102)	(0.0101)	(0.0100)	(0.0100)	(0.0100)	(0.0100)	(0.0100)
α	0.046097	0.109601	0.085881	0.082487	0.080652	0.083453	0.097108	0.102964
	(0.0045)	(0.0144)	(0.0134)	(0.0109)	(0.0101)	(0.0157)	(0.0152)	(0.0151)
τ	-0.12993	-0.18923	-0.2366	-0.22269	-0.21521	-0.22806	-0.21777	-0.20937
	(0.0414)	(0.0640)	(0.0668)	(0.0520)	(0.0475)	(0.0967)	(0.0795)	(0.0725)
β	0.946095	0.36206	0.603983	0.754384	0.79844	0.103492	0.191091	0.221113
	(0.0054)	(0.0573)	(0.0591)	(0.0329)	(0.0261)	(0.0920)	(0.0852)	(0.0776)
ω	0.003198	-2.8401	-3.97258	-4.84236	-5.28778	-2.98268	-3.18559	-3.24889
	(0.0009)	(0.1237)	(0.1915)	(0.2236)	(0.2546)	(0.1640)	(0.1689)	(0.1684)
δ_{IV}			15.28059	17.02478	18.09884	9.834606	9.53965	8.927864
			(0.9203)	(1.2311)	(1.4451)	(1.2288)	(1.3060)	(1.2911)
δ_{RV}		8.997185				4.540291	5.287555	5.878333
		(0.4734)				(0.7133)	(0.6696)	(0.6330)
LL	-3519	-3493	-3480	-3484	-3489	-3461	-3468	-3473
AIC	7048	6998	6972	6980	6989	6937	6951	6960
BIC	7079	7035	7009	7017	7026	6980	6994	7004

Table 5: QMLE for TGARCH models

			IV			RV-IV		
		RV	(1m)	(3m)	(6m)	(1m)	(3m)	(6m)
ψ	0.020777	0.010774	0.017692	0.017689	0.017832	0.015403	0.014896	0.01465
	(0.0098)	(0.0101)	(0.0100)	(0.0099)	(0.0099)	(0.0100)	(0.0100)	(0.0100)
α	0.029806	0.034567	0.026136	0.033296	0.037554	0.024736	0.032957	0.036884
	(0.0066)	(0.0170)	(0.0137)	(0.0111)	(0.0103)	(0.0198)	(0.0200)	(0.0199)
γ	0.032095	0.128975	0.108681	0.095921	0.086354	0.112355	0.121965	0.124845
	(0.0067)	(0.0230)	(0.0192)	(0.0159)	(0.0143)	(0.0252)	(0.0249)	(0.0248)
β	0.946001	0.465427	0.656795	0.752167	0.794148	0.124245	0.217311	0.254181
	(0.0055)	(0.0476)	(0.0544)	(0.0357)	(0.0289)	(0.0833)	(0.0811)	(0.0762)
ω	0.003968	-2.98205	-4.02199	-4.64056	-5.01743	-2.96525	-3.1658	-3.2353
	(0.0008)	(0.1070)	(0.1871)	(0.2053)	(0.2260)	(0.1509)	(0.1581)	(0.1612)
δ_{IV}			14.67783	16.07772	16.8247	9.472806	9.175149	8.563858
			(0.8965)	(1.1010)	(1.2532)	(1.2299)	(1.3175)	(1.3133)
δ_{RV}		8.741853				4.66929	5.35005	5.905651
		(0.4610)				(0.7159)	(0.6766)	(0.6423)
LL	-3517	-3490	-3477	-3481	-3487	-3460	-3467	-3471
AIC	7043	6991	6966	6975	6986	6933	6947	6957
BIC	7074	7028	7004	7012	7023	6977	6991	7000

Table 6: QMLE for AGARCH models

			IV			RV-IV		
		RV	(1m)	(3m)	(6m)	(1m)	(3m)	(6m)
ψ	0.029128	0.010598	0.020534	0.021208	0.021333	0.015987	0.015329	0.014993
	(0.0104)	(0.0104)	(0.0101)	(0.0102)	(0.0103)	(0.0102)	(0.0102)	(0.0103)
α_*	0.018522	0.050336	0.021001	0.023775	0.025087	0.041924	0.045324	0.046093
	(0.0004)	(0.0075)	(0.0096)	(0.0091)	(0.0087)	(0.0100)	(0.0089)	(0.0086)
β_*	1.001457	-0.00022	-0.4244	-0.33995	-0.32515	-0.01499	-0.0068	-0.00365
	(0.0001)	(0.0081)	(0.1656)	(0.1999)	(0.2247)	(0.0438)	(0.0291)	(0.0223)
ω_*	-0.00028	-2.23998	-2.51086	-2.73831	-2.81797	-2.78723	-2.85622	-2.86057
	(0.0002)	(0.0537)	(0.1346)	(0.1837)	(0.2024)	(0.1014)	(0.0998)	(0.1008)
δ_{IV}			16.02831	17.47311	18.05335	10.04036	9.45715	8.631713
			(0.6632)	(0.6911)	(0.6839)	(0.9444)	(0.9631)	(0.9283)
δ_{RV}		9.506536				4.642562	5.602547	6.315863
		(0.3865)				(0.5380)	(0.4820)	(0.4503)
LL	-3593	-3516	-3493	-3521	-3544	-3471	-3482	-3489
AIC	7195	7041	6996	7052	7099	6953	6976	6990
BIC	7220	7072	7027	7083	7130	6991	7013	7027

Finally, the EGARCH model has been successfully used to fit the returns of Latam exchange rates (Lafarguette and Veyrune, 2021; Jara and Piña, 2023). It describes the dynamics of the logarithm of the variance with an autoregressive process. However, there is only one shock in the model which can be understood as a perfect correlation between shocks in returns and volatility equations. Table 7 shows the results for several EGARCH models with exogenous variables (log-RV and log-IV) in the variance equation. The first columns shows that the model without exogenous variables has a high persistency in the log variance (98.5%). The overall fits, improves when either log-RV or log-IV are included; however, including both variables is preferred.

Table 7: QMLE for EGARCH models

			IV			RV-IV		
		RV	(1m)	(3m)	(6m)	(1m)	(3m)	(6m)
ψ	0.02162	0.00930	0.01804	0.01690	0.01718	0.01285	0.01199	0.01151
	(0.0094)	(0.0095)	(0.0098)	(0.0096)	(0.0096)	(0.0096)	(0.0096)	(0.0096)
κ_0	-0.01052	1.17293	2.97320	0.65361	0.44073	2.43089	2.07110	1.87343
	(0.0028)	(0.1748)	(0.4420)	(0.1014)	(0.0680)	(0.3766)	(0.3623)	(0.3332)
κ_1	0.98489	0.53956	0.18059	0.82261	0.88284	0.28879	0.39142	0.43958
	(0.0026)	(0.0647)	(0.1121)	(0.0253)	(0.0162)	(0.0981)	(0.0902)	(0.0839)
κ_2	0.02608	0.09176	0.07087	0.07001	0.05964	0.09158	0.09468	0.09456
	(0.0055)	(0.0150)	(0.0180)	(0.0111)	(0.0095)	(0.0172)	(0.0161)	(0.0156)
κ_3	0.11648	0.15437	0.19412	0.16980	0.16689	0.13672	0.15332	0.15812
	(0.0110)	(0.0240)	(0.0310)	(0.0213)	(0.0192)	(0.0308)	(0.0277)	(0.0267)
κ_{IV}			1.74102	0.38167	0.25663	0.76363	0.53058	0.40620
			(0.2536)	(0.0580)	(0.0385)	(0.1797)	(0.1486)	(0.1240)
κ_{RV}		0.79030				0.72316	0.74560	0.75785
		(0.1132)				(0.0948)	(0.1026)	(0.1074)
LL	-3506	-3447	-3461	-3473	-3481	-3430	-3437	-3440
AIC	7021	6906	6935	6959	6974	6873	6887	6894
BIC	7052	6944	6972	6996	7011	6916	6931	6937

2. SV Model

As we discussed in Section II, there are several alternatives to estimate the parameters of the model and we choose to use QMLE given the sample of this study. Table 8 shows the result for the SV model considering non-exogenous variables (first column), logarithm of the realized volatility (RV column), and logarithm of the implied volatilities at different tenors (IV columns). The model without exogenous variables has a high persistent (ϕ) and it attributes a small portion of the variance to the log volatility (η^2).[10] Moreover, the overall in-sample fit of the model can be assessed by considering the information criteria (AIC: Akaike, and BIC: Bayesian). Thus, the variable log-RV provides a significant improvement with respect to the baseline model, in contrast to the IV measures.

Table 9 shows the results for the SV model when the variance assumption of the dependent variable is removed. In the baseline model (without exogenous variables) the estimated variance of the measurement error is 4.201 which is lower than the one imposed in the standard model (4.935); including further explanatory variables to the log-variance equation the size of measurement errors reduces accordingly. Similar to the previous table, we can see a significant improvement of the overall fit of the model when the variable log-RV is included. These findings remain when the Fuller transformation is considered (see Appendix B) and when we consider correlation between measurement and the state errors.[11]

[10] In the table the constant c considers the actual constant of the model and the expected value of the $\log(u_t^2)$. A (non-reported) Monte Carlo exercise supports the accuracy of QMLE of the baseline model.

[11] Results that include correlation (not reported) are statistically non-significant.

Table 8: QMLE for SV model

	RV		IV		
			(1m)	(3m)	(6m)
ϕ	0.99281	0.21219	-0.20705	0.94548	0.97388
	(0.0028)	(0.1360)	(0.3340)	(0.0263)	(0.0123)
δ		1.56356	3.04053	0.13260	0.05790
		(0.2592)	(0.8502)	(0.0710)	(0.0343)
c	-0.99172	0.80508	1.42373	1.32955	1.11526
	(0.0924)	(0.0994)	(0.1306)	(0.2694)	(0.4409)
η^2	0.00809	0.00000	0.00000	0.02617	0.01636
	(0.0027)	(0.0000)	(0.0000)	(0.0118)	(0.0070)
LL	-4951	-4914	-4923	-4943	-4947
AIC	9908	9833	9851	9895	9902
BIC	9927	9852	9870	9919	9927

Table 9: QMLE for Extended SV model

	RV		IV		
			(1m)	(3m)	(6m)
ϕ	0.99029	0.25703	0.09230	0.91191	0.95680
	(0.0035)	(0.1434)	(0.3723)	(0.0408)	(0.0179)
δ		1.47447	2.28480	0.22020	0.10227
		(0.2741)	(0.9472)	(0.1110)	(0.0505)
c	-2.21080	1.78814	3.15876	3.09617	2.79372
	(0.1854)	(0.2134)	(0.2820)	(0.5245)	(0.7898)
η^2	0.01186	0.37136	1.49675	0.05434	0.03078
	(0.0039)	(0.3159)	(5.7805)	(0.0275)	(0.0131)
σ^2	4.20136	3.92208	2.83541	4.11853	4.14948
	(0.1029)	(0.3311)	(5.7287)	(0.1083)	(0.1049)
LL	-7822	-7788	-7798	-7810	-7815
AIC	15652	15586	15606	15630	15640
BIC	15677	15617	15637	15661	15671

V. Tail-Risk Framework

In this section we discuss how the models estimated above can be used to generate tail-risk indicators. We propose a general framework for doing this, and we provide an example.[12]

1. Framework

Following Lafarguette and Veyrune (2021), we build tail-risk statistics based on Monte Carlo simulations of the models presented before (GARCH / SV), calibrated for the CLP; however, we acknowledge that no-model is superior for this task. For that we propose the "MI framework" which involves the following steps:

- Step M: choose any of the models presented above (e.g., Taylor, Nelson, Heston-Nandi) along with an information set: (i) only returns, (ii) returns and IV, or (iii) returns and RV. Using previous results get the relevant parameters.
- Step I: choose a given horizon for tail-risk indicators (e.g., forecast density 1-day, 1-week, or 1-month ahead). Use MC simulations to get the relevant indicator if needed.

2. Applications

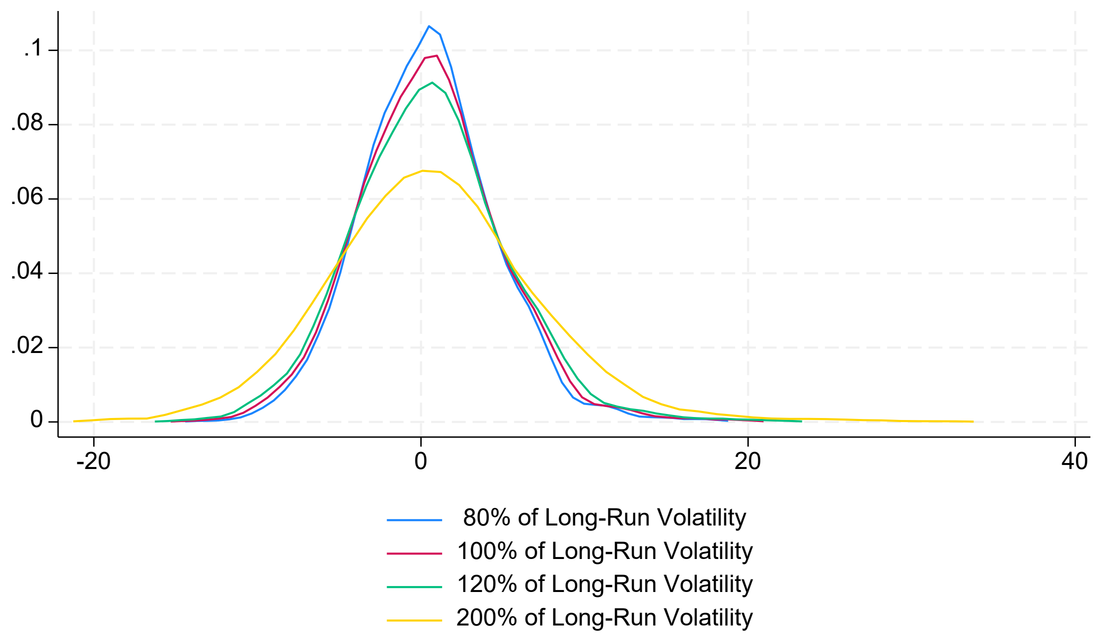
In our first application we have Step M: TGARCH model estimated only with returns for the sample 2008-2013 (see Table B.1). Step I: MC simulations for 5, 10, 15 and 30 days ahead, and percentiles 5th and 95th for four levels of initial volatility (80%, 100%, 120%, and 200%).[13] Assuming $\psi = 0$, we have $h_t = 0.048 + 0.086r_{t-1}^2 + 0.797h_{t-1} + 0.067r_{t-1}^2 d_{t-1}$, where d_t is a dummy variable that takes the value of one when r_t is positive and zero otherwise. Table 10 shows the results for 3000 simulations, where the cumulative returns on CLP over 30 days would be between -5.9% and 7.2% if the initial volatility is 80% of the long-run volatility (LRV), but the range increases to -6.8% and 8.3% if the initial volatility is 120% of LRV, and to -9.2% and 11.1% under 200% of LRV. Figure 5 shows the 30-day densities of the cumulative returns.

[12] It is important to note that we assume that a sufficient statistic is the time-variant volatility.
[13] A scenario of 200% of LRV materialized in 2022 for the case of IV 3-month where it reaches 25% and the average is 12% (Figure 4 & Table 2).

Table 10: Percentiles for the TGARCH model

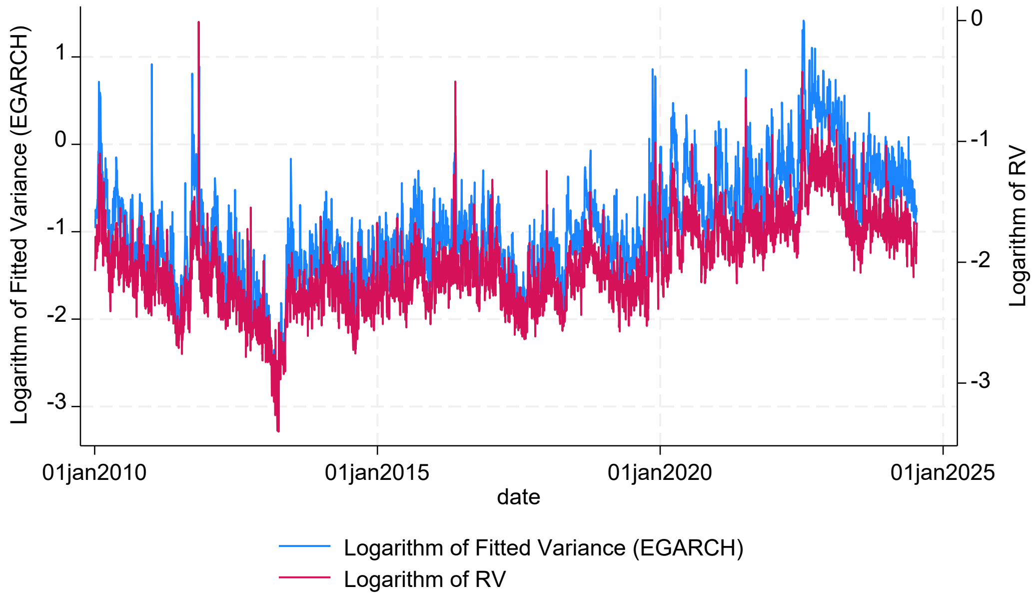
	80%		100%		120%		200%	
days	p5	p95	p5	p95	p5	p95	p5	p95
5	-2.232	2.430	-2.674	2.922	-3.118	3.421	-5.004	5.511
10	-3.139	3.789	-3.616	4.406	-4.146	5.047	-6.410	7.786
15	-4.077	4.662	-4.547	5.230	-5.145	5.897	-7.611	8.696
30	-5.854	7.153	-6.333	7.687	-6.784	8.282	-9.202	11.064

Figure 5: 30-day ahead densities for the TGARCH model



Our second application assumes (i) Step M: EGARCH model estimated with returns and the logarithm of RV using full sample (Table 7, second column). Thus, we have the following equation for variance: $\log(h_t) = 1.17 + 0.54\log(h_{t-1}) + 0.09z_{t-1} + 0.15\tilde{z}_{t-1} + 0.79\log(RV_{t-1})$, where $\tilde{z}_{t-1} \equiv |z_{t-1}| - \sqrt{2/\pi}$, and (ii) Step I: Forecasting 1-day ahead conditional variance and taking its logarithm. The correlation between the later and logarithm of RV is 87.2% and the dynamics are similar (Figure 6).

Figure 6: Logarithm of Conditional Variances



VI. Conclusions

In this paper we integrate an additional volatility indicator for the Chilean Peso (CLP) —intraday realized volatility (RV)— and the implied volatility (IV) into time-variant volatility models: GARCH and SV. Our empirical analysis, utilizing a comprehensive dataset of almost 14 years, spanning from 2010 to 2024, reveals that incorporating RV or IV significantly enhances the in-sample fit of both GARCH and SV models, improving their ability to capture the dynamics of CLP. This is consistent with the recent literature emphasizing the benefits of combining return and option-based indexes (Alfaro and Inzunza, 2023). Thus, our paper might hold significant economic implications for risk managers, and policymakers. Accurate tail-risk prediction enables better portfolio allocation strategies, more effective risk management practices within financial institutions, and more informed financial decisions. Specifically, our improved modelling of extreme scenarios can contribute to enhanced decision-making around hedging strategies, and financial responses to market shocks. Future research could explore the incorporation of additional macroeconomic variables (e.g., interest rate differentials, commodity prices), and investigate alternative model specifications.

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Appendix A: Technical Appendix

Gram-Charlier density: Jondeau and Rockinger (2001) report that this density is widely used in Finance, and it requires only the third and fourth moments to calibrate it. For the standardized returns (z_t) the density (type A) is:

$$f(z_t) = \phi(z_t)\left[1 + \frac{\gamma_1}{6}(z_t^3 - 3z_t) + \frac{\gamma_2}{24}(z_t^4 - 6z_t^2 + 3)\right]$$

where γ_1 and γ_2 are the skewness and the excess of kurtosis, respectively. There are some approximated expressions for these moments: (i) Duan et al. (1999) for GARCH model, (ii) Duan et al. (2006) for TGARCH and EGARCH models, (iii) Alexander et al. (2021) for TGARCH model with non-normal errors, and (iv) Mazzoni (2010) for Heston-Nandi model.[14] Alternatively, Monte Carlo simulations can be performed to obtain numerical figures as in Duan (1995).

Fuller's Transformation: Fuller (1996) provides an alternative for transforming the SV model into a linear state-space system. First, noting that $r_t^2 + \Delta = e^{m_t}u_t^2 + \Delta = e^{m_t}(u_t^2 + \Delta e^{-m_t})$, then we have: (i) $\log(r_t^2 + \Delta) = m_t + \log(u_t^2 + \Delta e^{-m_t})$, and (ii) $\Delta/(r_t^2 + \Delta) = \Delta e^{-m_t}/(u_t^2 + \Delta e^{-m_t})$. The author suggests considering (i) & (ii), which implies: $\log(r_t^2 + \Delta) - \Delta/(r_t^2 + \Delta) = m_t + w_t$, where $w_t = \log(u_t^2 + \Delta e^{-m_t}) - \Delta e^{-m_t}/(u_t^2 + \Delta e^{-m_t})$. This is because w_t (error term in the linear state-space system) is less skewed than $\log(u_t^2)$. Also, he suggests calibrating c as small fraction (for example, 2%) of the average value of r_t^2, which is equivalent to consider a small fraction of the sample variance of the returns (Abbara and Zevallos, 2019).

Intuition for Posteriors: consider a random variable $x_i = \mu + \sigma u_i$ with $u_i \sim N(0,1)$ and let $\bar{x}$ and s^2 be the average of n observations and the sample variance (adjusted by $n-1$). From finite-sample theory we have $\sqrt{n}\,(\bar{x} - \mu)/\sigma \sim N(0,1)$ and $n\,s^2/\sigma^2 \sim \chi_{n-1}^2$, and for asymptotic distributions we have $\sqrt{n}\,(s^2 - \sigma^2)/\sigma \sim N(0,2\sigma^4)$ and $\sqrt{n}\log(s^2/\sigma^2) \sim N(0,2)$. Thus, using a conventional uninformed prior for both mean (μ) and variance (σ^2) implies the following posteriors: normal distribution for mean $\mu \sim N(\bar{x}, \sigma^2/n)$ and inverse-gamma (IG) distribution for variance $\sigma^2 \sim IG[(n-1)/2\,;(n/2)s^2]$. Thus, an initial drawn of σ^2 is needed for getting a drawn

[14] For GARCH, TGARCH, and EGARCH models formulae involve lengthy sums; meanwhile for Heston-Nandi model, the proposal of Mazzoni (2010) offers a closed-form expression.

of μ which posterior is consistent with the result presented above. Further, the inverse-gamma distribution has two relevant properties. For $M \sim IG[a; b]$ we have (i) $kM \sim IG[a; kb]$, and (ii) $M \sim IG[a; 1/2] = Inv\chi^2_{2a}$ (inverse χ^2). Using (i) we have $\sigma^2/(ns^2) \sim IG[(n-1)/2\,; (1/2)]$, and by taking (ii) we have $\sigma^2/(ns^2) \sim Inv\chi^2_{n-1}$ which implies $n\, s^2/\sigma^2 \sim \chi^2_{n-1}$, given the properties of the χ^2 distribution. In other words, the posteriors for mean and variance based on uninformed priors are consistent with the finite-sample results presented above.

Appendix B: Additional Estimates

Jara and Piña (2023) use a two-state Markov Switching GARCH model to fit the returns on CLP for the period 1999-2020. From their results, a "calm" regimen was observed in 2013-2017 and a "turbulent" one during the global financial crisis period (2008-2012). Similar to Lu and Qu (2022), we use these two samples to estimate the parameters of the GARCH models.

Table B.1 shows the results of GARCH, NGARCH y TGARCH for the two samples described above. Some comments about it:

i. parameter estimates are consistent within a regimen, but they differ between samples; for example, β is around 80% in turbulent times (columns 1 to 3) and around 93% during calm periods (columns 4 to 6),
ii. the asymmetry parameter (τ) is similar in the two samples being around 9%, but the threshold parameter (γ) is somehow different, as overall the persistent parameter (λ) which combinates all previous parameters with α being lower during the calm periods, this implies a smooth updating during no-stressed periods in contrast with the stressed ones, and, as expected,
iii. the annualized long-run volatility ($15.81 \cdot \sigma_{LR}$) is 12.2% for the turbulent times and 10.5% for the calm periods; however, the standard error is significantly higher for the latter due to the high degree of persistent,
iv. the estimates for the average daily returns are not statistically different than zero.

Table B.2 extends the results presented in Section IV by considering the Fuller transformation (see Appendix A). Based on these we can confirm that the measure RV provides a significant improvement in the overall fit of the model compared to IV measures.

Table B.1: QMLE for GARCH models

		2008-2012			2013-2017	
	GARCH	NGARCH	TGARCH	GARCH	NGARCH	TGARCH
ψ	-0.02284	-0.01729	-0.01513	0.02266	0.02664	0.02545
	(0.0193)	(0.0207)	(0.0202)	(0.0152)	(0.0155)	(0.0157)
α	0.12700	0.12138	0.08583	0.06192	0.05661	0.04555
	(0.0196)	(0.0205)	(0.0235)	(0.0104)	(0.0107)	(0.0151)
β	0.79554	0.79675	0.79728	0.92872	0.93545	0.93495
	(0.0255)	(0.0266)	(0.0271)	(0.0127)	(0.0127)	(0.0128)
τ		-0.09408			-0.09021	
		(0.0638)			(0.0663)	
γ			0.06721			0.02344
			(0.0269)			(0.0176)
ω	0.04687	0.04768	0.04869	0.00411	0.00310	0.00347
	(0.0061)	(0.0065)	(0.0066)	(0.0018)	(0.0017)	(0.0016)
λ	0.92254	0.91813	0.91671	0.99063	0.99206	0.99222
	(0.0133)	(0.0139)	(0.0136)	(0.0067)	(0.0057)	(0.0060)
σ	12.30	12.20	12.09	10.47	10.59	10.56
	(0.61)	(0.57)	(0.55)	(1.90)	(2.02)	(2.09)
LL	-1379.0	-1378.1	-1376.8	-1047.5	-1046.5	-1046.8
AIC	2766.0	2766.2	2763.6	2102.9	2103.0	2103.6
BIC	2786.5	2791.8	2789.2	2123.4	2128.6	2129.2

Table B.2: QMLE for Extended SV model (Fuller Transformation)

		RV		IV	
			(1m)	(3m)	(6m)
ϕ	0.99003	0.27748	0.03080	0.91399	0.95376
	(0.0036)	(0.1373)	(0.0183)	(0.0341)	(0.0178)
δ		1.26494	2.15541	0.18952	0.09763
		(0.2316)	(0.1144)	(0.0824)	(0.0448)
c	-2.03978	1.48842	2.70473	2.63885	2.42430
	(0.1610)	(0.1780)	(0.2335)	(0.4477)	(0.6598)
η^2	0.00944	0.24076	2.99985	0.03976	0.02492
	(0.0030)	(0.1924)	(0.6714)	(0.0170)	(0.0097)
σ^2	2.89389	2.72507	0.00000	2.83581	2.85357
	(0.0712)	(0.2056)	(0.6744)	(0.0738)	(0.0724)
LL	-7153	-7119	-7129	-7141	-7146
AIC	14315	14249	14268	14292	14302
BIC	14340	14280	14299	14323	14333

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