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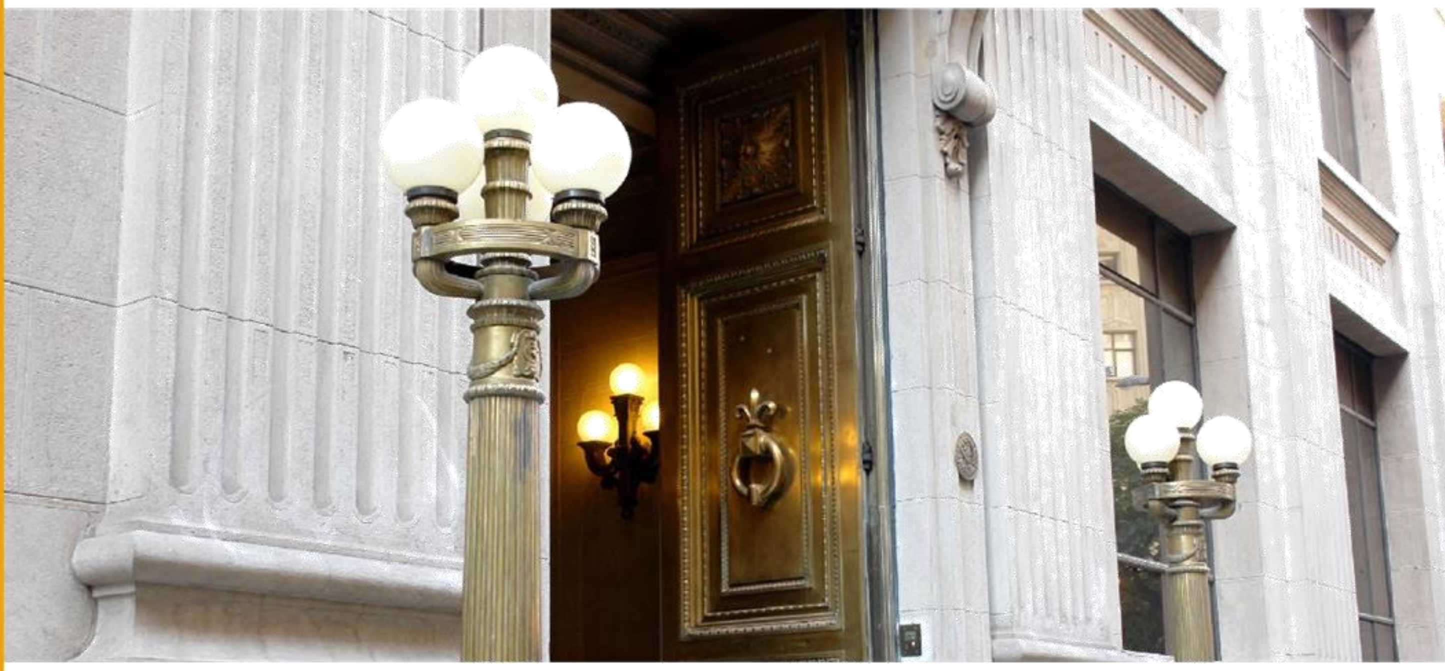

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Modeling S&P500 returns with GARCH models*

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Abstract

This paper provides several estimates of the parameters of GARCH models for the S&P500 index, based on returns and CBOE VIX. Using a daily sample from 2007 to 2022, we conclude that adding the information of VIX improves the estimates of the long-term volatility. We provide an external validation of the model using an option-based index reported by the Federal Reserve of Minneapolis, allowing us to propose a calibrate model for tracking the tail-risk of this stock index.

Resumen

Este documento proporciona estimaciones de los parámetros de modelos GARCH para el índice S&P500, basados en rendimientos y en el CBOE VIX. Usando una muestra diaria de 2007 a 2022, concluimos que agregar la información de VIX mejora las estimaciones de la volatilidad de largo plazo. Brindamos una validación externa del modelo utilizando un índice basado en opciones informado por la Reserva Federal de Minneapolis, lo que nos permite proponer un modelo de calibración para cuantificar el riesgo de cola en este índice bursátil.

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I. Introduction

GARCH models were introduced by Engle (1982) and then widely used in financial econometrics for empirical modeling of asset prices (Singleton, 2002), as they offer a powerful tool for characterizing the distribution of asset prices, which oftentimes requires Monte Carlo simulations to obtain relevant statistics such as tail-risk measures. Duan (1995) established that a GARCH model can also be used to value options by introducing the so-called locally risk-neutral valuation relationship (LRNVR), in which a simple algebraic manipulation is needed to change the probability measure. For instance, if the returns can be modeled with a standard GARCH in mean, which characterizes the physical measure, then option prices must be evaluated using a non-linear GARCH model, which characterizes the risk-neutral measure. The extension to other GARCH models is straightforward if the disturbances are normally distributed; moreover, approximations formulae are already published for several cases (eg., Duan et al., 1999; Heston and Nandi, 2000; Hao and Zhang, 2013; Kannianen et al., 2014). Thus, the shape of these density functions, under both the physical and risk-neutral measures, depends on the GARCH's parameters as well as the initial level of conditional variance. Since conditional variance is also estimated with the model, the GARCH's parameters are crucial for proper estimation of density and probability functions.

Hao and Zhang (2013) and Kannianen et al. (2014) pointed out that the VIX, which is a well-known volatility index generated by the Chicago Board Option Exchange (CBOE) using option prices, can be obtained in closed form from several GARCH models. Therefore, they both proposed to include it as an additional variable to estimate GARCH's parameters; nonetheless, reaching opposite conclusions. On one side, Hao and Zhang (2013) argued that the joint estimation generates "unreasonable parameters", and that the implied VIX from the GARCH model is not in line with the observed one in “various statistical aspects”. On the other, Kanniainen et al. (2014) stated that joint estimates “improve option pricing performance”, compared with other methods such as direct calibration. Zhang and Zhang (2020) tried to close that discrepancy by modifying the LRNVR, yet the data does not support their approach.

In this paper, we conducted several empirical strategies for estimating the parameters of two GARCH models, considering daily data of the S&P500 returns and VIX from January 2007 to December 2022, combining both series with different weights. This extends previous papers which consider equal weights for the returns and the VIX. Also, we discuss alternative ways to introduce the VIX information, by defining the error term with the VIX squared, level of the VIX, and the logarithm of the VIX. Based on these we conclude that: (i) the parameter associated with the risk-premium increases from 9% (only-returns) to 18% - 27% (depending on the function) when the VIX information is included, using a similar weight than returns. Point estimates are higher when only the VIX information is included. These findings are in line with Hao and Zhang (2013) who used a sample between January 1990 and August 2009; however, extended to a more recent sample, and using different functions to incorporate the VIX information (level, squares, logarithms). This does not imply that the model with information of VIX generates a higher tail-risk, because other parameters of the model are also involved in such measure, but it agrees that unreasonable values are obtained when the information of the VIX is weighted at 100%. And (ii) the estimate of the unconditional volatility (long-term volatility) falls when the information of VIX is included. This result is robust when we consider the level of VIX, VIX squared or the logarithm of VIX. Thus, our point estimate with only-returns implies 21% long-term volatility (annual terms), while by using the logarithm of VIX (or the level of VIX) it stands at around 16%, and 18% when the VIX-squared is considered. These findings differ from Hao and Zhang (2013) and Kannianen et al. (2014), as these authors found that the unconditional volatility increases when the VIX information is included. Interestingly, estimates of this parameter seem to be “reasonable values” even when the information of VIX is weighted at 100%. A zoom-in shows that the reduction occurs rapidly when the information of the VIX is included, already recognizable at 5% weight.

In order to study the adequacy of our GARCH models, we use a tail-risk indicator provided by the Federal Reserve of Minneapolis (FRM); which approximates a Large Decrease Probability (LDP) in the stock market index over the next 6 months (20% fall). Thus, we simulate LDP using our GARCH models, finding that Threshold GARCH (TGARCH) provides a better fit to the data when we consider the VIX information in logarithm. Still, there is room for improvement in the model, that is something that we leave for future developments.

The article is developed as follows. In Section 2, we introduce the GARCH models used in this research, along with the LRNVR, the closed-form expression for the VIX, and the Monte Carlo approach for other option-based indexes (eg., LDP). In Section 3, we provide estimates of the parameters of the GARCH and TGARCH models based on a daily sample from 2007 to 2022, a comparison between VIX and the model implied ones, and an exercise using the LDP series. In Section 4, we have a policy discussion and Section 5 concludes.

II. Model Setup

In this section we discuss: (i) the key elements of our GARCH in-mean models and the locally risk-neutral valuation relationship (LRNVR), (ii) the relationship between the parameters of our GARCH models and the VIX, and (iii) the Monte Carlo simulations approach to compute other option-based indexes.

1. GARCH models

In the GARCH models the log-return of the asset, which is the log-difference of the asset price (P_t), has the following dynamic:

$$\Delta \log P_t = r - \frac{1}{2} h_t + \lambda \sqrt{h_t} + \sqrt{h_t} u_t, \tag{1}$$

where r is the risk-free short-term interest rate, h_t is the conditional variance, and u_t is — under the physical measure— an independent Gaussian error term (zero mean unit variance). Conditional on h_t, the log-return for the next period is normally distributed; however, for short or medium run the probability distribution function of the cumulated log-returns departures from Gaussian distribution. The equation includes one-half of the conditional variance, which is the Jensen term, due to the use of logarithmic returns. It also includes the term $\lambda\sqrt{h_t}$ is the risk-premium component, which is the size of excess returns investors require as risk compensation.

The conditional variance is defined recursively as:

$$h_t = \omega + \alpha h_{t-1} u_{t-1}^2 + \beta h_{t-1} + \gamma h_{t-1} u_{t-1}^2 (u_{t-1} < 0), \tag{2}$$

where ω is related with the level of the unconditional variance and it is assumed to be positive, α controls for the degree of heteroscedasticity and it is also assumed to be positive, meaning that when $\alpha = 0$ the equation collapses to a deterministic expression which implies a constant variance, β is used for assessing the degree of persistence of the conditional variance, and it is assumed to be positive. Finally, γ captures the impact of negative shocks in the conditional variance; thus, we name GARCH model when this is parameter is imposed to be zero and Threshold GARCH (TGARCH) otherwise.[1] For further reference, we define $\theta = (\lambda, \omega, \alpha, \beta, \gamma)'$ as the vector of parameters.

It is important to note that h_{t+1} is known at time t, and given that u_t is standard normal we could write the following expression: $E_t(P_{t+1}) = P_t \exp\left(r + \lambda\sqrt{h_{t+1}}\right)$, where the conditional expectation is computed under the physical measure using all the information available up to time t. Thus, the parameter λ plays a role in controlling the degree of compensation that investors required for holding this risky asset. It should be noted that the compensation is expressed in terms of the volatility in the GARCH/TGARCH models, but it could be defined in other forms. The idea of additional term was introduced by Engle et al. (1987) with the purpose of having a time-variant risk premium. The author suggested to use the volatility but recognized that other functions, such as the logarithm of the variance, have better fit.[2] Heston and Nandi (2000) proposed a model based on the conditional variance with a particular specification of the dynamic of that variable. These

[1] There are alternative functions for the conditional variance, therefore other GARCH models (StataCorp, 2019). For example, the exponential-GARCH (EGARCH) defines the dynamics of the conditional variance model in logarithms. We leave other GARCH specifications for future research.
[2] "The choice of the standard deviation represents the assumption that changes in variance are reflected less than proportionally in the mean. Empirically, the log of h(t) is found to be even better." (Engle et al., 1987)

assumptions allowed them to have a closed form expression for the European call options using a similar argument than Heston (1993).[3]

2. The LRNVR assumption

Duan (1995) introduced the locally risk-neutral valuation relationship (LRNVR) which allow us to value options using the GARCH/TGARCH models. If we define $v_t = u_t + \lambda$, and assuming that follows a standard Gaussian disturbance under the risk-neutral measure, then using (1) we have: $\Delta \log P_t = r - 0.5h_t + \sqrt{h_t}v_t$. The expected value of the asset price, under the risk-neutral measure, is $r - 0.5h_t$, which implies that asset price is a discounted martingale ensuring that there is not arbitrage opportunities. This change in the measure, also affects the conditional variance: $h_t = \omega + \alpha h_{t-1}(v_{t-1} - \lambda)^2 + \beta h_{t-1} + \gamma h_{t-1}(v_{t-1} - \lambda)^2(v_{t-1} < \lambda)$, adding to that a non-linear asymmetric component in line with Table 1 of Engle and Ng (1993).[4] After some algebra, the equation of the conditional variance, under the risk-neutral measure, can be expressed as follows $h_t = \omega + \rho h_{t-1} + \alpha h_{t-1}e_{t-1}$, where e_{t-1} is an error term with zero mean and finite variance, and ρ is the persistence of the conditional variance. Following, Duan et al. (2006) and given that disturbances are normally distributed we have $\rho = \alpha(1 + \lambda^2) + \beta + \gamma[\lambda\phi(\lambda) + (1 + \lambda^2)\Phi(\lambda)]$, where $\phi(\quad)$ and $\Phi(\quad)$ are, respectively, the density and cumulative functions of the standard normal distribution. Assuming $0 < \rho < 1$, we can also obtain the unconditional variance ($\bar{h}$) as:

[3] Given these assumptions the generating function is closed form, then the option price can be expressed in terms of that; however, the procedure requires to solve an integral over the complex space. Several alternatives have been proposed to speed up that part; for example, Mazzoni (2010) suggested an approximation of the generating function and uses that to obtain the skewness and kurtosis in closed form. Along with these terms, the author proposed use them into a Gram-Charlier density function.

[4] It should be noted that Engel and Ng (1993) did not specify the return equation of the GARCH models; thus, some of the specification of the risk premium could be not consistent with the conditional variance equation, in the sense that some parameters are not identified.

$\bar{h} = \omega/(1-\rho)$. Note that the conditional expectation of h_{t+2}, has the following expression: $E_t^*(h_{t+2}) = \bar{h}(1-\rho) + \rho h_{t+1}$, where $E_t^*(\ )$ is the conditional expectation computed under the risk-neutral measure using all the information available up to time t. Iterating this expression, we have the following formula: $E_t^*(h_{t+k}) = \bar{h}(1-\rho^{k-1}) + \rho^{k-1} h_{t+1}$.

In summary: (i) under the physical measure the volatility term in the return equation implies that the model is GARCH/TGARCH in mean, and (ii) under the risk-neutral measure the GARCH/TGARCH model imposes an expected value according to the risk-free rate, but it implies a change in the conditional variance equation.

3. The CBOE Volatility Index (VIX)

The VIX is an index provided by the Chicago Board Options Exchange, based on the squared root of a weighted average of one-month option prices. The goal of the index is to approximate the expected value of the variance swap; thus, its interpretation should be in squared terms. Given that, the relationship between the daily variance based on the VIX (X_t) and the conditional variance (h_{t+1}) is (Kanniainen et al., 2014):

$$X_t \equiv \frac{1}{\tau}\left(\frac{VIX_t}{100}\right)^2 \cong \frac{1}{n}\sum_{k=1}^{n} E_t^*(h_{t+k}), \qquad (3)$$

where n stands for the maturity of the VIX and τ for the annualizing parameter. We use n = 22 for trading days, and τ = 365 according to VIX calendar count convention. The conditional expectation is computed under the risk-neutral measure. It can be obtained in closed form for the

GARCH/TGARCH models presented above, plugging the expression obtained in the previous section, we have:

$$X_t = \left[1 - \frac{1-\rho^n}{n(1-\rho)}\right]\bar{h} + \left[\frac{1-\rho^n}{n(1-\rho)}\right]h_{t+1}. \tag{4}$$

For the empirical section we consider the following error term: $w_t = g(X_t) - g(A + Bh_{t+1})$, with A and B according to (4) and $g(\quad)$ a given function (identity, squared root or logarithm).

4. The Federal Reserve of Minneapolis (FRM) indexes

The FRM computes several indexes based on option-prices with 6- and 12-months expirations, including skewness, kurtosis, and probabilities of extreme events. The procedure is based on Shimko (1993) and Malz (1997) in the sense that the available option prices are summarized into implied-volatilities and strike-price space. A cubic spline is used to characterize that relationship which is then used to generate call prices. Finally, indexes are computed in accordance with statistical formulae (eg., skewness and kurtosis) or by taking second derivative of call prices (eg., risk-neutral density).

In contrast to the VIX, the option-based indexes obtained from FRM do not have a closed form equation that relate them with the parameter of the GARCH or for the TGARCH model. Duan et al. (1999) provided formulae for the first four moments of the GARCH model, meanwhile Hongwiengjan and Thongtha (2021) provided similar expressions for the TGARCH model.[5] For

[5] Duan et al. (2006) provided those for the EGARCH model.

our implementation, we follow Duan (1995), by approximating the expectation with I Monte Carlo replications of the asset-price process:

$$E_t^*[f(P_{t+n})] = \frac{1}{I}\sum_{i=1}^{I} f\big(P_{(i),t+n}\big), \tag{5}$$

where $P_{(i),t+n}$ is the terminal value $(t + n)$ of the i-simulated path, and $f()$ is an arbitrary function. For the tail-risk, we compute the probability value of being below a given threshold or the Large Decrease Probability (LDP). Following the ones published by the FRM we choose a threshold value of 80%, equivalent to a 20% loss.

III. Empirical Analysis

In this section we describe the variables used in the estimation. Then, discuss the procedure and the results obtained under different setups, comparing the results implied by the models with both the CBOE VIX and the FRM Large Decrease Probability (LDP).

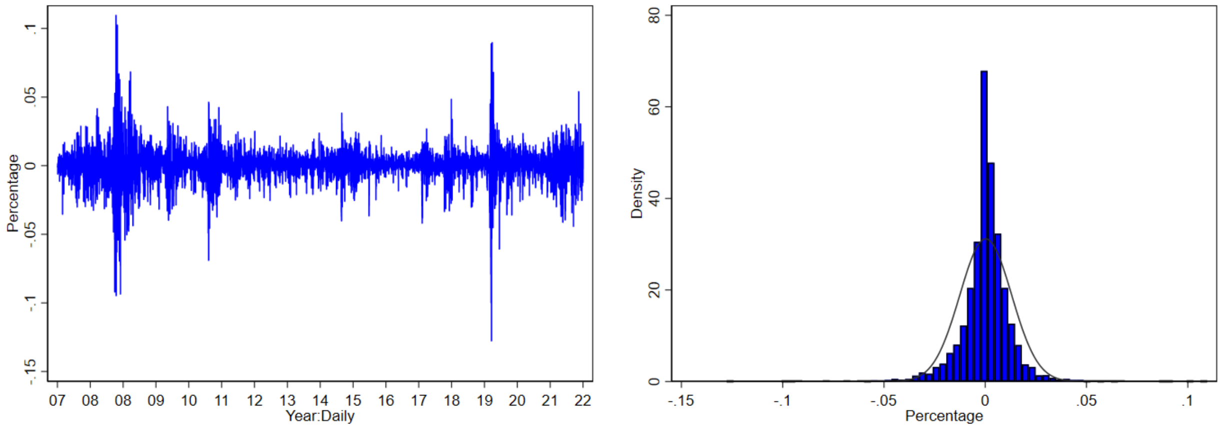
1. Data

The time-series data used to estimate the GARCH models are the daily closing prices of the S&P500, VIX and LDP, ranging from January 2, 2007, to December 30, 2022. For the daily risk-free interest rate, we use the three-month (secondary market) Treasury bill rate retrieved from the Federal Reserve Bank of St. Louis website. These series have 4,175 daily observations available for study.[6]

We obtain the continuously compounded returns of the S&P500 series after log-differencing, which are portrayed in Figure 1 (left). The daily mean of the series is 0.02% with a standard deviation of 0.013, it exhibits a negative skewness of -0.521 and kurtosis of 15.344. In Figure 1 (right) we also provide the density distribution of returns, with a fitted normal (black line). A simple inspection suggests that returns do not follow a normal distribution, thus a GARCH model seems to be appropriate for returns.

[6] For non-trading days related to stock market holidays, we compute the value of the last trading day available.

Figure 1. S&P 500 returns

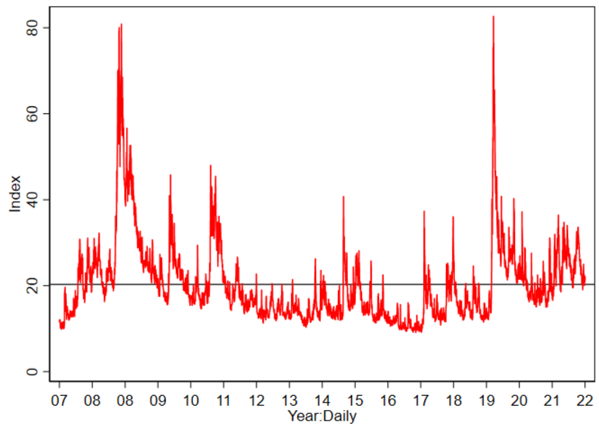


Source: Bloomberg

Figure 2.a. presents the VIX series, which is calculated from a portfolio of S&P500 options with maturity of one month and associated with perceived risk, with easily noticeable spikes at and after the 2008 financial crisis, and at the beginning of 2020 in the surge of the COVID-19 pandemic. The mean of the series is at 20.28 (black line) with a standard deviation of 9.3. Figure 2.b. displays the LDP for 6 months, these estimates were retrieved from its website, available for the January 2007 – December 2022 period, with 383 biweekly observations for study. The biweekly mean for the sample period is at 9.11% with a standard deviation of 4.67%. A simple inspection suggests that a key component in LDP is the VIX, thus a measure of conditional variance can be used for fitting the LDP measure.

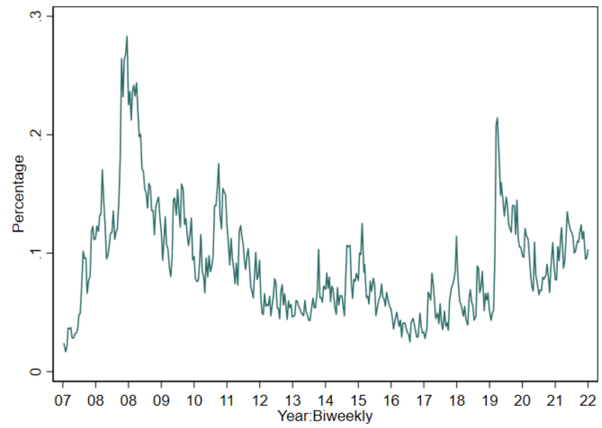
Figure 2. Option-based Indexes for S&P500

a. CBOE VIX and mean



Source: Bloomberg

b. FRM LDP



Source: Federal Reserve of Minneapolis

2. Joint Likelihood

To estimate the GARCH model parameters we use the maximum likelihood (ML) approach, for returns and VIX. For the returns data, the loglikelihood function $\log L_{RET}$ is:

$$\log L_{RET}\,(\theta) = -\frac{1}{2}\sum_{t=2}^{T}\left[\log h_t + \frac{\left(\Delta \log P_t - r_t + 0.5h_t - \lambda\sqrt{h_t}\right)^2}{h_t}\right], \tag{6}$$

and the corresponding function for the VIX data is:[7]

$$\log L_{VIX}\,(\theta) = -\frac{1}{2}\left[(T-1)\log s^2 + \frac{1}{T}\sum_{t=2}^{T}\frac{w_t^2}{s^2}\right]. \tag{7}$$

[7] Hao and Zhang (2013), Kanniainen et al. (2014), and Zhang and Zhang (2020) choose g() as the squared root function, but we also consider the identity and the logarithm functions for completeness. Thus, we have results for the squared of VIX, the level of VIX, and the logarithm of VIX.

where $w_t = g(X_t) - g(A + Bh_{t+1})$ and s^2 is the variance of w_t. Thus, the joint likelihood on returns and VIX could be defined as a linear combination of both log-likelihood functions:

$$\log L_{TOT}(\theta) = (1 - \kappa) \log L_{RET}(\theta) + \kappa \log L_{VIX}(\theta) \quad (8)$$

where κ is the parameter that controls the weight assigned to the VIX data. Hao and Zang (2013), Kanniainen et al. (2014), and Zhang and Zhang (2020) consider three cases ($\kappa = 0$, $\kappa = 0.5$ and $\kappa = 1$), which are associated to only returns, returns and VIX (equally weighted), and only VIX. We extend these choices by running the optimization problem with several values of κ.

3. Main Results

Table 1 provides a selected set of parameters for the GARCH models weighting-in the returns and VIX series and imposing $\gamma = 0$, with its corresponding standard errors in parentheses; first we provide estimates using only returns, then we increase the relative importance of the VIX (by rising κ) and show different combinations of weights and VIX types (squares, levels, and logarithms).

As we put more relative weight into the VIX data compared to returns, we yield an increase in the risk-premium parameter (λ), that moves from 9% (column 1) to 187% (column 3) which implies that the adjusted GARCH model (under the risk-neutral measure) has a high degree of asymmetry; however, this is compensated by a decreasing heteroscedasticity parameter (α) that moves from 15% to 2%. The increase in the risk-premium parameter is in line with the results of previous papers, furthermore it comprises an increase in the persistence parameter (ρ) under the risk-neutral measure. These findings are somehow robust for different functions applied to the VIX; however, the use of logarithms implies a smooth transition in the parameter estimates as we will discuss next.

Table 1. ML estimates for GARCH models using returns and VIX
(daily data, January 2007 – December 2022)

Column	Returns	Returns & VIX Squared		Returns & VIX		Returns & Log VIX	
κ	0	0.5	1	0.5	1	0.5	1
λ	0.0931	0.2656	1.8709	0.2090	1.2711	0.1767	0.6137
	(0.0147)	(0.0214)	(0.0228)	(0.0258)	(0.0402)	(0.0256)	(0.0398)
ω	2.89.E-06	2.92.E-06	2.31.E-06	2.17.E-06	1.77.E-06	1.63.E-06	1.45.E-06
	(2.43.E-07)	(1.28.E-07)	(4.76.E-08)	(7.85.E-08)	(3.37.E-08)	(7.75.E-08)	(3.64.E-08)
α	0.1455	0.0432	0.0243	0.0504	0.0287	0.0639	0.0452
	(0.0090)	(0.0006)	(0.0002)	(0.0012)	(0.0004)	(0.0025)	(0.0012)
β	0.8369	0.9267	0.8722	0.9248	0.9089	0.9178	0.9241
	(0.0092)	(0.0008)	(0.0014)	(0.0015)	(0.0020)	(0.0028)	(0.0016)
ρ	0.9836	0.9730	0.9818	0.9775	0.9839	0.9836	0.9864
	(0.0043)	(0.0003)	(0.0002)	(0.0005)	(0.0003)	(0.0010)	(0.0005)
$\sqrt{h}$	0.2109	0.1651	0.1788	0.1557	0.1664	0.1586	0.1639
	(0.0244)	(0.0032)	(0.0015)	(0.0022)	(0.0012)	(0.0027)	(0.0018)
VIX^{Mean}	21.706	20.511	20.837	20.176	20.321	20.301	20.289

Note: This table displays the Maximum likelihood estimates obtained from the GARCH model, using returns, VIX or both. In parentheses are standard errors. Persistence, long-term volatilities, and the mean of the model implied VIX are also reported.

The parameters for the TGARCH models ($\gamma \neq 0$) are provided in Table 2; like the previous table, we start by using only returns, then we increase the relative importance of the VIX and show different combinations of weights and VIX types. All columns show that parameter γ is significant, but before choosing between GARCH or TGARCH model, we will discuss which function of VIX seems to be more appropriate.

As with the GARCH model, the TGARCH yields an increase in the risk-premium parameter (λ), showing that equal-weights ($\kappa = 0.5$) provides reasonable estimates for this parameter. For example, when we consider returns only it stands at 4.9%, while for VIX^2 only it reaches 243% and for the case $\kappa = 0.5$ implies 16%. Similar to GARCH model results, the use of the logarithm of the VIX seems to be more appropriate, we will extend this argument a little bit further.

Table 2. ML estimates for TGARCH models using returns and VIX
(daily data, January 2007 – December 2022)

Column	Returns	Returns & VIX Squared		Returns & VIX		Returns & Log VIX	
κ	0	0.5	1	0.5	1	0.5	1
λ	0.0490	0.1638	2.4342	0.1125	1.0152	0.0832	0.4642
	(0.0160)	(0.0245)	(0.0143)	(0.0289)	(0.0589)	(0.0288)	(0.0545)
ω	2.84.E-06	1.85.E-06	2.34.E-06	1.73.E-06	1.52.E-06	1.49.E-06	1.38.E-06
	1.98.E-07	(8.98.E-08)	(4.83.E-08)	(6.74.E-08)	(3.14.E-08)	(6.94.E-08)	(3.50.E-08)
α	0.0218	-0.0242	0.4336	-0.0034	-0.0221	0.0198	0.0252
	(0.0046)	(0.0011)	(0.0203)	(0.0025)	(0.0031)	(0.0051)	(0.0040)
β	0.8575	0.9554	0.8298	0.9471	0.9273	0.9334	0.9326
	(0.0077)	0.0012	0.0013	0.0015	0.0025	0.0026	0.0018
ϒ	0.1903	0.0790	-0.4118	0.0631	0.0530	0.0542	0.0235
	(0.0126)	(0.0013)	(0.0203)	(0.0036)	(0.0034)	(0.0068)	(0.0053)
ρ	0.9821	0.9815	0.9814	0.9812	0.9860	0.9842	0.9866
	(0.0038)	(0.0003)	(0.0001)	(0.0004)	(0.0003)	(0.0008)	(0.0005)
$\sqrt{h}$	0.2003	0.1586	0.1782	0.1522	0.1656	0.1544	0.1608
	(0.0241)	(0.0033)	(0.0015)	(0.0022)	(0.0014)	(0.0027)	(0.0020)
VIX^{Mean}	21.71	20.51	20.84	20.18	20.33	20.30	20.29

Note: This table displays the Maximum likelihood estimates obtained from the TGARCH model, using returns, VIX or both. In parentheses are standard errors. Persistence, long-term volatilities, and the mean of the model implied VIX are also reported.

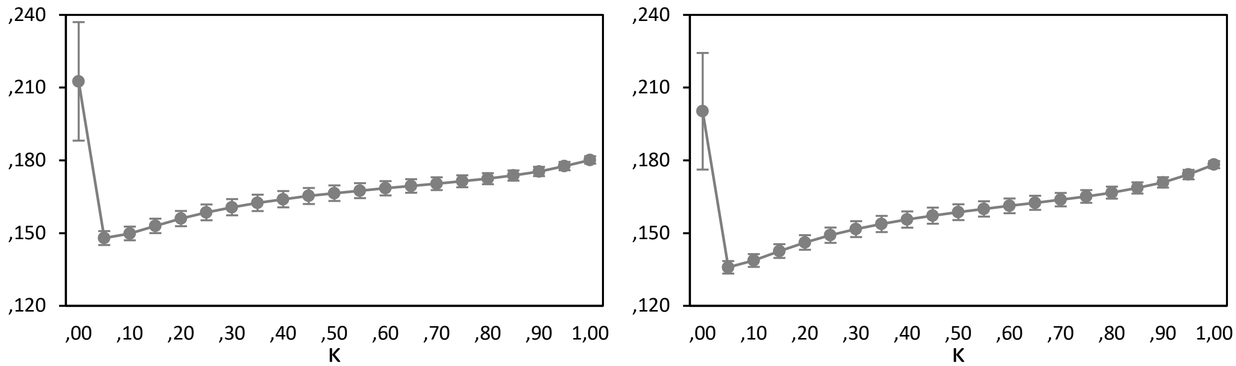
Figure 3 shows the estimates of the long-term volatility (LTV) at different values of κ, for both the GARCH and TGARCH models, with their respective confidence intervals.[8] We observe that weighting a little bit the information of the VIX reduces the LTV, then when more weight is considered, these estimates increase again but the LTV remains significant below the cases when only-returns are used for the estimation.[9]

[8] Noticeably, by increasing the importance of the VIX information, we yield more precise estimates of both measures. This is in line with previous papers, and it is derived by the fact that adding the information of the VIX we have a similar effect that "increasing" the sample size.
[9] Indeed, when $\kappa = 0.05$ we have LTV of 14.8% using VIX squared and the GARCH model, same occurs for other VIX functions and for the TGARCH model.

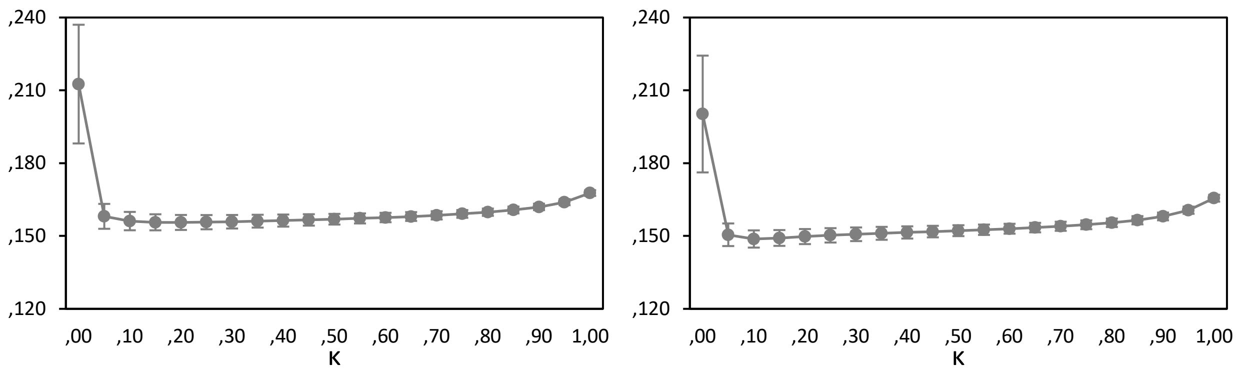
Figure 3. Estimated long-term volatility using returns and VIX

a. GARCH b. TGARCH

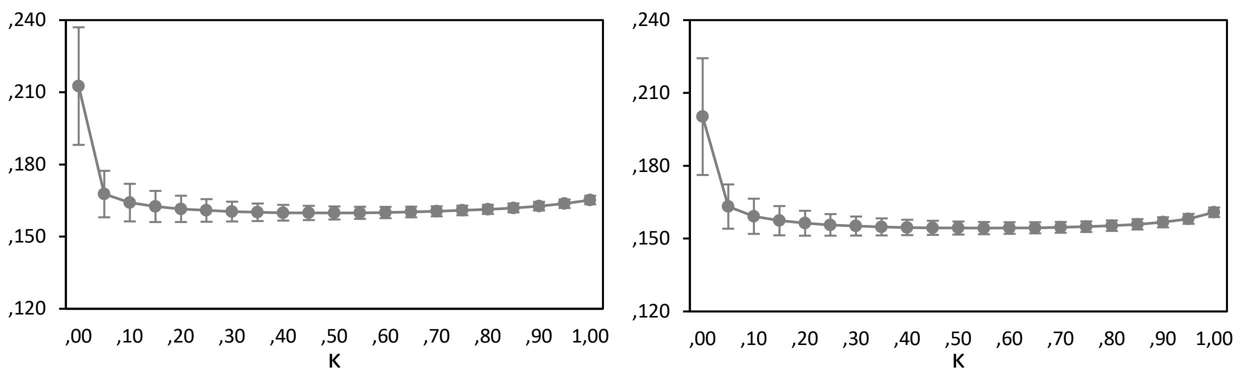
I. VIX Squared



II. Level of VIX



III. Logarithm of VIX

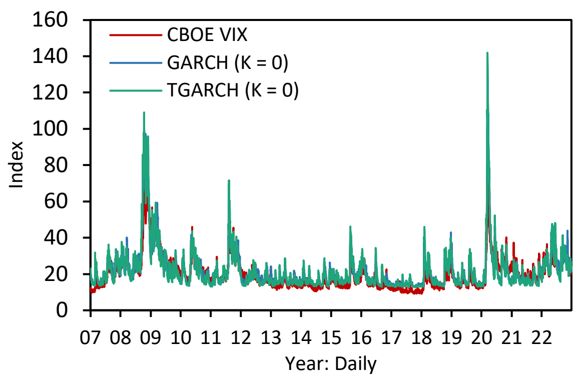


Note: This figure displays the estimated long-term volatility (LTV) obtained from the GARCH and TGARCH model, at different VIX weights, and VIX functions; and provides the 95% confidence intervals for the estimates.

Using the parameters in Tables 1 and 2, we estimate the model implied VIX according to (4) and compare it to the original VIX series. This is portrayed in Figure 4, with the returns only case in panel a, and VIX functions in panels b to d, the latter panels all include equally weighted returns and VIX (κ = 0.5) for ease of comparison. The correlation among the VIX series and the implied ones are displayed under each figure, while the corresponding descriptive statistics are in Table 3 and Table 4. Figure 4 shows that for both models the parameters obtained from equally weighting returns and the logarithm of VIX yield the highest correlation between the series, additionally both models yield similar results, in line with previous papers.

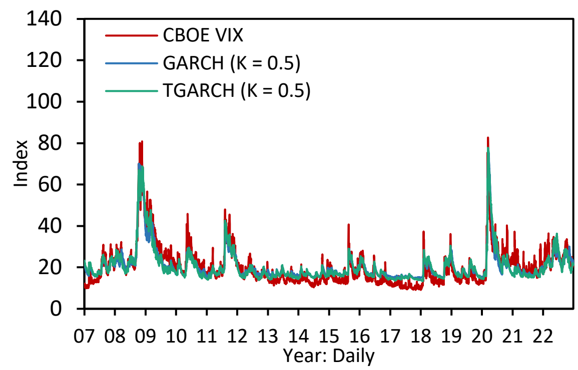
Figure 4. Implied VIX compared to CBOE VIX

a. Returns



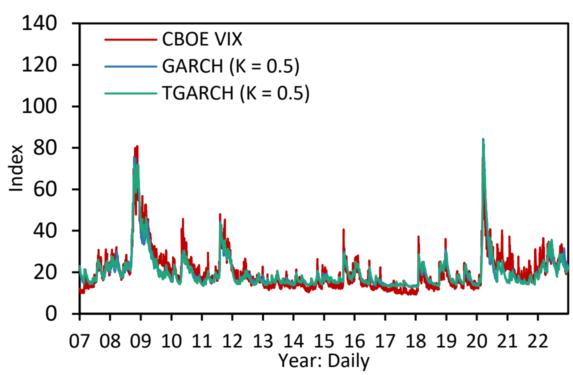
corr = 0.8977 / 0.8972

b. VIX Squared



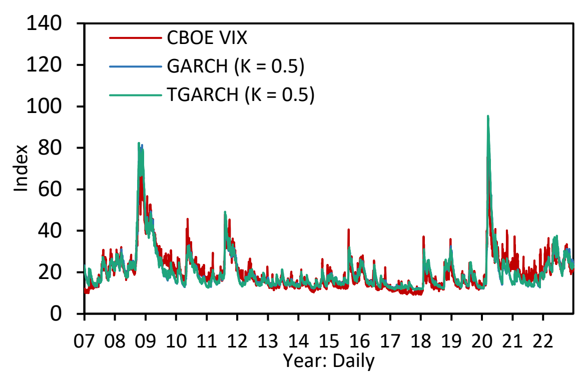
corr = 0.9099 / 0.9170

c. Level of VIX



corr = 0.9131 / 0.9218

d. Logarithm of VIX



corr = 0.9250 / 0.9231

Note: This figure displays GARCH and TGARCH models implied VIX by equally weighing-in returns and VIX, against the CBOE VIX. Under each figure we report the correlation between the series, first between GARCH implied VIX and CBOE VIX, second between the latter and TGARCH implied VIX.

Depending on the chosen model and VIX function, the implied VIX better fits certain aspects of the CBOE VIX distribution and its statistical properties. As shown in Table 3, the logarithm of VIX produces an implied VIX with the least difference between the original series in terms of its mean, skewness and kurtosis, and the lower part of the distribution. On the other hand, the TGARCH model (Table 4) using the logarithm of VIX function, closely matches the mean, the lowest part of the distribution and the 90th percentile.

Table 3. GARCH implied VIX distribution compared to CBOE VIX

	CBOE VIX	a. Returns	b. VIX Squared	c. VIX	d. Log VIX
Mean	20.281	21.888	20.645	20.130	**20.281**
SD	9.297	11.345	7.772	**8.864**	10.221
Skewness	2.291	3.825	3.700	3.549	**3.291**
Kurtosis	7.759	20.336	17.157	16.033	**13.851**
AR1	0.977	0.980	0.994	0.994	0.993
AR10	0.865	0.807	0.899	0.898	0.890
AR30	0.666	0.480	0.608	0.609	0.597
Min	9.140	12.711	14.707	12.950	**11.525**
p10	12.160	14.235	15.680	14.216	**13.149**
p25	13.950	15.388	16.474	15.239	**14.450**
Median	17.840	18.519	**18.242**	17.408	17.087
p75	23.700	**24.219**	22.012	21.956	22.705
p90	30.608	**31.599**	27.075	27.783	29.342
Max	82.690	131.238	77.681	**84.379**	89.223

Note: This table displays the GARCH model implied VIX and compares it to CBOE VIX, in terms of its distribution and statistical properties. The bold values indicate the least percentage difference compared to the original series.

Table 4. TGARCH implied VIX distribution compared to CBOE VIX

	CBOE VIX	a. Returns	b. VIX Squared	c. VIX	d. Log VIX
Mean	20.281	21.706	20.511	20.176	**20.301**
SD	9.297	11.822	8.204	**8.922**	10.281
Skewness	2.291	3.801	**3.223**	3.299	3.321
Kurtosis	7.759	20.691	**12.867**	13.686	14.283
AR1	0.977	0.974	0.991	0.993	0.993
AR10	0.865	0.770	0.900	0.904	0.895
AR30	0.666	0.445	0.659	0.649	0.620
Min	9.140	12.734	13.227	12.766	**11.623**
p10	12.160	13.795	14.946	14.107	**13.110**
p25	13.950	14.883	15.889	15.152	**14.386**
Median	17.840	17.982	**17.762**	17.301	17.121
p75	23.700	**24.315**	22.111	22.064	22.706
p90	30.608	32.387	27.671	27.737	**29.284**
Max	82.690	142.007	77.764	**83.187**	95.461

Note: This table displays the GARCH model implied VIX and compares it to CBOE VIX, in terms of its distribution and statistical properties. The bold values indicate the least percentage difference compared to the original series.

As result of this comparison, the use of the logarithm of the VIX seems to be more appropriate in both models: GARCH and TGARCH, but with the former having a slightly higher correlation (between implied VIX and actual VIX). To complete the diagnostic, the Schwarz criteria of these models are -19683 -19801, respectively. Thus, for the overall fit (returns and VIX) the TGARCH model is more appropriate, and it is still good enough for approximating the VIX.

4. External Validity Exercise

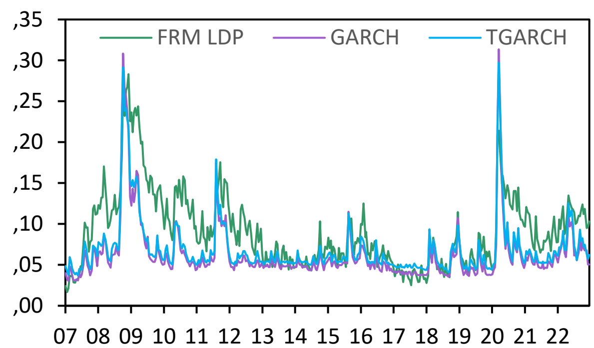
In this section, we compare the probabilities of a 20% decrease using the parameters obtained from different weights of the logarithm of VIX (as it better fits the VIX), against the LDP series. Figure 5 provides our models implied probabilities, Table 5 and Table 6 describe the distributions and properties.[10] A visual inspection suggests that there is room for improvement regarding the model

[10] For each implied LDP we use actual returns and the risk-free interest rate described in Section III.1 along with 2000 Monte Carlo simulations.

used to approximate the FRM LDP series, but the highest correlation between the FRM LDP and the model implied 20% LDP is obtained using TGARCH and VIX only information ($\kappa = 1$), noting that GARCH model offers a slightly lower correlations in all cases (Figure 5). Moreover, the TGARCH model (when $\kappa = 1$) more closely fits the mean, skewness and kurtosis of the original series, as well as the 25th, 50th, 75th, and 90th percentile of its distribution (Table 6). Thus, the TGARCH model seems to be more appropriated.

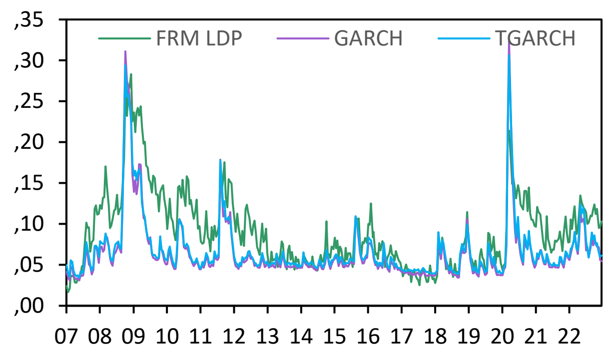
Figure 5. Estimated tail probabilities compared to FRM LDP

a. $\kappa = 0.25$



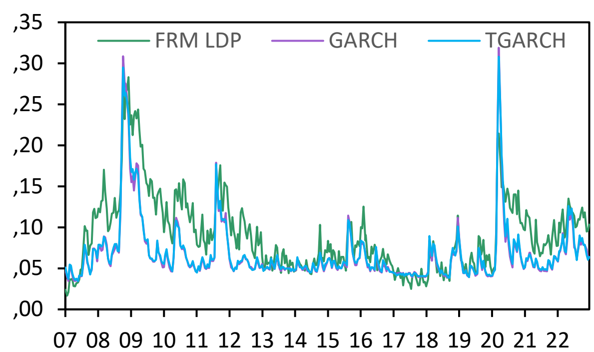
corr = 0.791 / 0.800

b. $\kappa = 0.50$



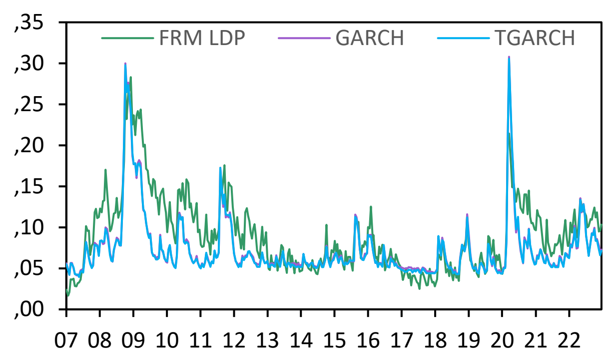
corr = 0.807 / 0.819

c. $\kappa = 0.75$



corr = 0.816 / 0.826

d. $\kappa = 1.00$



corr = 0.832 / 0.835

Note: This figure displays GARCH and TGARCH models implied 20% decrease probability by weighing-in different combinations of returns and VIX, against the FRM LDP. Under each figure we report the correlation between the series, first between GARCH implied probabilities and FRM LDP, second between the latter and TGARCH implied probabilities.

Table 5. GARCH estimated probability descriptives compared to FRM LDP

	FRM LDP	$\kappa = 0.25$	$\kappa = 0.50$	$\kappa = 0.75$	$\kappa = 1.00$
Mean	0.091	0.063	0.065	0.068	**0.074**
SD	0.047	0.037	0.039	**0.039**	0.039
Skewness	1.340	3.831	3.660	3.501	**3.248**
Kurtosis	2.350	18.003	16.315	14.764	**12.607**
Min	0.017	**0.031**	0.031	0.035	0.042
p10	0.043	0.040	0.039	**0.043**	0.050
p25	0.057	0.046	0.047	0.048	**0.054**
Median	0.080	0.053	0.055	0.056	**0.062**
p75	0.116	0.065	0.070	0.072	**0.079**
p90	0.147	0.097	0.100	0.106	**0.112**
Max	0.283	0.314	0.324	0.319	**0.308**

Note: This table displays the GARCH model obtained tail probabilities and compares it to the Large Decrease probability informed by the Federal Reserve of Minneapolis, in terms of its distribution and statistical properties. We highlight the values with the least percentage difference compared to the original series.

Table 6. TGARCH estimated probability descriptives compared to FRM LDP

	FRM LDP	$\kappa = 0.25$	$\kappa = 0.50$	$\kappa = 0.75$	$\kappa = 1.00$
Mean	0.091	0.069	0.069	0.069	**0.073**
SD	0.047	0.035	0.038	0.039	**0.039**
Skewness	1.340	3.553	3.369	3.298	**3.193**
Kurtosis	2.350	15.314	13.627	12.925	**12.130**
Min	0.017	0.037	0.035	**0.035**	0.040
p10	0.043	0.047	0.043	**0.043**	0.048
p25	0.057	0.052	0.050	0.049	**0.052**
Median	0.080	0.060	0.058	0.059	**0.061**
p75	0.116	0.072	0.073	0.074	**0.079**
p90	0.147	0.099	0.103	0.107	**0.113**
Max	0.283	**0.298**	0.306	0.308	0.306

Note: This table displays the TGARCH model obtained tail probabilities and compares it to the Large Decrease probability informed by the Federal Reserve of Minneapolis, in terms of its distribution and statistical properties. We highlight the values with the least percentage difference compared to the original series.

IV. Policy Discussion

In the previous sections we show that, based on the work of Duan (1995), GARCH/TGARCH models can be used to combine two sources of information: returns, and option-prices (VIX); this has a practical application because, in general, there are several indicators for a given financial market and therefore our proposal is to use GARCH/TGARCH models as a framework for combining these inputs. In our empirical section, we extended previous results available in the literature by: (i) updating the sample (2007-2022), (ii) considering different ways to incorporate the information of VIX (level, squares, logarithms), and (iii) by using different weights in the joint likelihood. Thus, our results can be used as robustness tests for the GARCH/TGARCH models that we examined. Finally, we use an additional option-based indicator reported by the Federal Reserve of Minneapolis in order to provide an external validation of the models: the Large Decrease Probability. Based on all these exercises we conclude that a proper way to combine the information is to consider the logarithm of the VIX and to use the TGARCH model as a framework for filtering the data related to the S&P 500. A similar approach can be applied to any other financial market.

V. Conclusions

GARCH models combined with Monte Carlo simulations are useful tools for modelling asset prices, which can in turn be used to compute tail-risk measures. We use information of log-returns, and VIX to estimate GARCH models. Our empirical results are in line with previous papers in the area (Hao and Zhang, 2013; Kanniainen et al., 2014; Zhang and Zhang, 2020) in terms of including VIX information, and contribute by broadening this approach with different combinations of VIX and returns. We propose a calibrate TGARCH model for computing tail-risk measures that matches current estimates of large decreases, although there is room for improvement. Our framework can be applied to other financial markets. We are considering this for future developments as well as other GARCH models such the EGARCH.

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