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A strategic analysis of "Expectations and the neutrality of money"*

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Abstract

"Expectations..." is the first counter-example to the view that a positive correlation between real output and the growth rate of the stock of money is exploitable. The equilibrium concept is rational expectations equilibrium. Here, two alternative strategic formulations -two versions of the market-game model- are applied. In one, the young make non-contingent offers of real saving; in the other, they make contingent offers, where the contingency is the realization of the two shocks in the model. Under the informational assumption that the young know nothing about current realizations, neither strategic formulation converges under replication to an equilibrium that exhibits the above positive correlation.

Resumen

"Expectations and the neutrality of money" ("Expectativas y la neutralidad del dinero", en castellano) es el primer contraejemplo a la postura según la cual se puede explotar una correlación positiva entre el producto real y la tasa de crecimiento del acervo de dinero. El concepto de equilibrio que usa es el de equilibrio de expectativas racionales. En este artículo se aplican dos formulaciones estratégicas alternativas: dos versiones del modelo de juegos de mercado. En una, los jóvenes hacen ofertas no contingentes de ahorro real; en la otra, hacen ofertas contingentes, donde la contingencia es la realización de las dos variables aleatorias en el modelo. Bajo el supuesto de información de que los jóvenes no saben nada sobre las realizaciones de las variables aleatorias, ninguna de las formulaciones estratégicas converge bajo réplicas a un equilibrio que exhiba la correlación positiva.

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1 Introduction

Lucas (1972) is a well-known and, arguably, the first counter-example to the view that a positive correlation between real output and the growth rate of the stock of money is exploitable—is invariant to the process generating the stock of money. We argue that it is a questionable counterexample. His equilibrium concept is rational-expectations equilibrium, which is widely regarded as problematic because it is nonstrategic. We analyze the model using two alternative strategic formulations—two versions of the *market-game* model.[1] One version has the young in his OLG model make non-contingent offers of real saving. The other has them make contingent offers, where the contingency is the realization of the two shocks in the model—a real aggregate-supply shock in the form of the number of young agents and a nominal aggregate-demand shock in the form of a proportional transfer of money to the old.[2] Under Lucas's informational assumption that the young know nothing about current realizations (other than their joint distribution), neither strategic formulation converges under replication to an equilibrium that exhibits the above positive correlation. Instead, they converge to other allocations.

In the game with non-contingent offers, we study three alternatives regarding what the young know when they choose an amount of output to save. We always assume that the young know the model and the pre-transfer total quantity of money, but do not know the monetary shock, the current transfer to the old. The three alternatives regarding what the young know are: (i) the young know nothing about the current shocks; (ii) the young know the ratio of the two shocks; (iii) the young know the aggregate-supply shock. Under each, the game has a unique active trade stationary equilibrium that approaches a limit under replication. Under (i), that limiting equilibrium has constant per capita output (the C-allocation). Under (ii), that limiting equilibrium is the one on which Lucas focussed (the L-allocation). Under (iii), that limiting equilibrium has output dependent only on the real supply shock (the N-allocation). In the game with contingent offers, no matter what the young know, the N-allocation is the unique active-trade stationary limiting equilibrium (see Table 1).

Thus, we get the L-allocation, the allocation that gives the positive correlation between output and the growth rate of the stock of money, only in the non-contingent

[1]We use the variant of the Shapley-Shubik trading-post game in Shubik (1973).

[2]Narayana Kocherlakota suggested that we analyze the contingent-offer version.

Table 1: Active-trade, stationary, limiting equilibrium allocations

	what the young know		
	nothing	ratio of shocks	supply shock
non-contingent market-game	C	L	N
contingent market-game	N	N	N

offer version of the game and only when the young know the ratio of the shocks. The general way to interpret the assumption that the young see the ratio of the shocks is that they see an informative and imperfect signal of both shocks. However, in the set of all such imperfect signals, the ratio is a very special signal.[3]

2 The environment

The model is a one-market version of Lucas (1972) with the smallest finite supports for the shocks that allow for the kind of partial information setting in that paper—a special case of the setting in Wallace (1992). There are two-period lived overlapping generations and there is one good per date. The integer size of a generation is denoted N where $N \in \{N_l, N_h\}$ and where $N_h > N_l \geq 2$. (When we replicate, we replace N by $kN \in \{kN_l, kN_h\}$ and let $k \to \infty$.) Money comes into the system by way of proportional transfers to each old person, so that if a person when young acquired m amount of money, each offers when old $(1+\gamma)m$ amount of money where $\gamma \in \{\gamma_l, \gamma_h\}$, and where

$$(1+\gamma_l)/N_l = (1+\gamma_h)/N_h > 0. \tag{1}$$

Both N and γ are distributed uniformly and independently of each other and over time.[4]

Each young agent is endowed with $w > 0$ amount of the good when young and nothing when old, has information I, and chooses real saving $x \in [0, w]$. Throughout, it is assumed that the young person knows the pre-transfer total quantity of money,

[3] The suggestion that the model might be interpreted as one in which the young see the ratio of the shocks appears in the exposition of the model in Stokey, Lucas and Prescott (1989, page 541).

[4] In Lucas (1972), the supports for the shocks are intervals. That eliminates the need for the assumption in (1) or its general analogue in Wallace (1992). The assumption in (1) is needed if we treat the number of people as an integer. We do that because we want to analyze a game with a finite number of players and want to use replication to draw conclusions about what happens in a large economy.

but not the realization of γ. The payoff to a young agent is

$$P(x) = u(w - x) + E_I v(Rx), \tag{2}$$

where R, an endogenous random variable, is the real return on x, and E_I denotes expectation conditional on I. As in Lucas (1972), the functions u and v are strictly increasing, strictly concave, twice differentiable, and satisfy $u'(0) = v'(0) = \infty$. Moreover,

$$-\frac{v''(z)z}{v'(z)} \in [a, 1) \quad \text{for some positive } a \tag{3}$$

The upper bound in (3) ensures that consumption when young and consumption when old are gross substitutes, whereas the lower bound is a mild strengthening of concavity of v.

Total measured GDP at a date is taken to be Nx. The simplest interpretation is that w is an endowment of the good and that only sales of the good for money, x, appear as part of measured GDP. Alternatively, w can be viewed as an endowment of leisure which can be transformed one-for-one into output. Then $w - x$ is leisure consumed, which does not appear in measured GDP, and x is production, which does appear in measured GDP.

3 A *market-game* with non-contingent offers

At each date, there is a trading post at which money trades for the good at that date. We model the post as a simultaneous-move game in which the actions of the participants are quantities offered. The money-side is trivial; each old person offers all their money. The good-side of the market has each young person offering an amount of the good. We let x denote the real saving of a person and let x_- denote the average real savings of the other members of the current cohort. Then, the person who saves x acquires

$$m(x, x_-) = \frac{M}{(N-1)x_- + x}x \tag{4}$$

amount of money, where N is the size of the current cohort and M is the post-transfer total stock of money. Then, at the next date, the person who played x will offer all their money, $(1 + \gamma')m(x, x_-)$, where γ' is the proportional transfer to all old people at the next date. Letting x_+ denote the average real savings of the members of the next cohort, consumption when old is

$$x' = \frac{N'x_+}{(1+\gamma')M}(1+\gamma')m(x,x_-) = \frac{N'x_+}{(N-1)x_- + x}x, \tag{5}$$

where the second equality follows by substitution from (4) and where N' is the size of the next cohort. Finally, because the real return R in $P(x)$ is defined to be x'/x, we get

$$R = \frac{N'x_+}{(N-1)x_- + x}. \tag{6}$$

This formula for R reflects the fact that the ultimate trade in this model of two-period lived overlapping generations is goods supplied by the current cohort of young for goods supplied by the next cohort of young. It is random because N, N' and x_+ are, in general, random from the point of view of the person choosing x.

Note that m in (4) and hence R in (6) is not well-defined for $x_- = x = 0$. When such is the case, we define m and R to be zero. Because we focus on equilibria in which such choices are never optimal, we can safely ignore this case for the remainder of the text.

When we replicate, R becomes

$$R_k = \frac{kN'x_+}{(kN-1)x_- + x} = \frac{N'x_+}{(N-1/k)x_- + x/k}. \tag{7}$$

Then we have the following definition of an equilibrium of the market-game with non-contingent offers.

Definition. *For a given k and a given specification of the information known by the young, $x = x_- = x_+ = \hat{x}$ is a symmetric, stationary, active-trade Nash equilibrium if $\hat{x} > 0$ and $\hat{x} = \arg\max_{x\in[0,w]} P(x)$ when R in $P(x)$* (see (2)) *is given by* (7).

In this definition, x, x_-, and x_+ are each functions, whose domain varies with what the young know about the current realizations of (N,γ). When the young know nothing about current realizations of the shocks, they are scalars and

$$P(x) = u(w-x) + \frac{1}{4}\sum_N \sum_{N'} v\left(\frac{N'x_+}{(N-1/k)x_- + x/k}x\right).$$

In this case, the equilibrium condition reduces to a single equation, the first-order condition, $\partial P(x)/\partial x = 0$, evaluated at $x = x_- = x_+$. When the young know the ratio of the current shocks, the domain is the three-element set of such ratios, $\{(1+\gamma_l)/N_h, (1+\gamma_l)/N_l, (1+\gamma_h)/N_l\}$, and the equilibrium condition is three first-order conditions at equality, one for each element in that set. These are simultaneous equations because all the components of x_+ appear in the support of R_k. When the young know N, the domain is the two-element set $\{N_l, N_h\}$ and the equilibrium condition is two first-order conditions—which, again, are simultaneous equations.

Implicit in this definition of an equilibrium is the assumption that each current generation knows nothing about the saving decision of the previous generation. If they did see those actions, then x_+ would have to be modeled as a function of those actions. One virtue of the assumption that each current generation knows nothing about the saving decision of the previous generation is that it rules out punishment strategies of the sort described in Kandori (1992), strategies that would seem to make the use of money superfluous.

Proposition 1. *Fix the information of the young at any one of the three specifications introduced above and fix* k*. (i) There exists a unique symmetric, stationary, active-trade equilibrium,* $\hat{x}_k$*. (ii) The limit of* $\hat{x}_k$ *as* $k \to \infty$ *exists and is a symmetric, stationary, active-trade equilibrium.*

Because most of the details follow arguments in Lucas (1972), they are given in the appendix. Here, we simply outline the main ingredients of each part of the argument.

Fix the information of the young and k and let p be the cardinality of the domain of $\hat{x}$ corresponding to that information structure. Define the choice problem of a young agent for a given $(x_-, x_+) \in \mathbb{R}^{2p}_{++}$. That gives rise to a strictly concave objective. Moreover, even though we allow the agent to choose zero saving, the unique optimal choice is positive and satisfies a system of first-order conditions. That allows us to apply the technique used by Lucas to establish the existence of the solution to the first-order conditions for a given k. More specifically, we use a transformation of the first-order conditions to obtain a mapping T whose fixed points after being transformed coincide with the solutions of the first-order conditions. Under assumption (3), that mapping is shown to be a contraction mapping $\mathbb{R}^p$ to itself. Because $\mathbb{R}^p$ is complete, the contraction mapping theorem can be invoked to get a unique fixed point.

Moreover, T is shown to be a contraction uniformly over $1/k$ for values of $1/k$ in the unit interval. In addition, because T is continuous in $1/k$ we are able to invoke Proposition 3.4.5 in Krantz and Parks (2012, page 50) to establish that the mapping from values of $1/k$ to the fixed points of T is continuous. It follows that as $1/k$ approaches zero—i.e., as k approaches infinity—the solutions converge to a stationary, active-trade equilibrium.

Although there may be an equilibrium with $x = x_- = x_+ = 0$, the above argument does not produce that equilibrium because we face the agent with $(x_-, x_+) \in \mathbb{R}^{2p}_{++}$ and obtain a fixed point in $\mathbb{R}^p$, which when transformed gives positive trade.[5]

In order to describe the limiting allocations in Table 1, we describe the limiting (as $k \to \infty$) real return distributions. First, we note that

$$\lim_{k\to\infty} R_k = \lim_{k\to\infty} \frac{N' x_+}{(N - 1/k)x_- + x/k} = \frac{N' x_+}{N x_-} \tag{8}$$

which, of course, corresponds to having each young agent take the real-return distribution as given. Each specification of what the young know implies a different conditional distribution of $N' x_+ / N x_-$. All are uniform distributions, so we only need to describe the supports evaluated at $x_+ = x_- = \hat{x}$.

If the young know nothing, then $\hat{x}$ is a scalar and the limiting support is

$$\left\{ \frac{N_l}{N_h}, \frac{N_l}{N_l}, \frac{N_h}{N_h}, \frac{N_h}{N_l} \right\}. \tag{9}$$

The C-allocation is optimal saving when a young person faces a uniform return distribution over that support.

If the young see the ratio $\rho = (1 + \gamma)/N$, then there are three limiting conditional supports, one for each of the three magnitudes of ρ: $\rho_l = (1+\gamma_l)/N_h$, $\rho_m = (1+\gamma_l)/N_l = (1 + \gamma_h)/N_h$, and $\rho_h = (1 + \gamma_h)/N_l$, with corresponding $\hat{x} = (\hat{x}_l, \hat{x}_m, \hat{x}_h)$. Using (8), the conditional supports are:

$$\left\{ \frac{N_l \hat{x}_m}{N_h \hat{x}_l}, \frac{N_l \hat{x}_h}{N_h \hat{x}_l}, \frac{N_h \hat{x}_l}{N_h \hat{x}_l}, \frac{N_h \hat{x}_m}{N_h \hat{x}_l} \right\} \text{ if } \rho = \rho_l, \tag{10}$$

$$\left\{ \frac{N_l \hat{x}_m}{N_h \hat{x}_m}, \frac{N_l \hat{x}_h}{N_h \hat{x}_m}, \frac{N_h \hat{x}_l}{N_h \hat{x}_m}, \frac{N_h \hat{x}_m}{N_h \hat{x}_m}, \frac{N_l \hat{x}_m}{N_l \hat{x}_m}, \frac{N_l \hat{x}_h}{N_l \hat{x}_m}, \frac{N_h \hat{x}_l}{N_l \hat{x}_m}, \frac{N_h \hat{x}_m}{N_l \hat{x}_m} \right\} \text{ if } \rho = \rho_m, \tag{11}$$

[5] A different kind of argument would follow the argument in Wallace (1992) and apply Brouwer's fixed point theorem at $1/k = 0$ to a set that is bounded away from zero as in Manuelli (1986). That would not require assumption (3). Then, a genericity argument could be invoked to allow the implicit function theorem to be applied in a neighborhood around $(1/k) = 0$. That would give Proposition 1 for all sufficiently large k.

and

$$\left\{\frac{N_l\hat{x}_m}{N_l\hat{x}_h}, \frac{N_l\hat{x}_h}{N_l\hat{x}_h}, \frac{N_h\hat{x}_l}{N_l\hat{x}_h}, \frac{N_h\hat{x}_m}{N_l\hat{x}_h}\right\} \text{ if } \rho = \rho_h. \tag{12}$$

The L-allocation is such that each component of $\hat{x} = (\hat{x}_l, \hat{x}_m, \hat{x}_h)$ is optimal saving for the respective observation of ρ.

If the young see the current realization of N, then there are two conditional supports:

$$\left\{\frac{N_l\hat{y}_l}{N_l\hat{y}_l}, \frac{N_h\hat{y}_h}{N_l\hat{y}_l}\right\} \text{ if } N = N_l \tag{13}$$

and

$$\left\{\frac{N_l\hat{y}_l}{N_h\hat{y}_h}, \frac{N_h\hat{y}_h}{N_h\hat{y}_h}\right\} \text{ if } N = N_h, \tag{14}$$

where in this case we denote the corresponding savings by $\hat{x} = (\hat{y}_l, \hat{y}_h)$. The N-allocation is such that each component of $\hat{x} = (\hat{y}_l, \hat{y}_h)$ is optimal saving for the respective observation of N.

Notice that the monetary transfer rates, (γ_l, γ_h), appear only when the young see the ratio of the shocks and only as conditioning information. Hence, a correlation between that growth rate and total output can appear only in that case. In that case, the conditional distributions of total output are:

$$\text{if } \gamma = \gamma_l, \text{ then } Nx = \begin{cases} N_l\hat{x}_m \text{ with prob } 1/2 \\ N_h\hat{x}_l \text{ with prob } 1/2 \end{cases}, \tag{15}$$

and

$$\text{if } \gamma = \gamma_h, \text{ then } Nx = \begin{cases} N_l\hat{x}_h \text{ with prob } 1/2 \\ N_h\hat{x}_m \text{ with prob } 1/2 \end{cases}. \tag{16}$$

Following the argument in Lucas (1972), it is shown in Wallace (1992) that $\hat{x}_l < \hat{x}_m < \hat{x}_h$. That and (15) and (16) imply that the correlation between γ and Nx is positive. (The linear regression of Nx on γ goes through the points $[\gamma_l, (N_l\hat{x}_m + N_h\hat{x}_l)/2]$ and $[\gamma_h, (N_l\hat{x}_h + N_h\hat{x}_m)/2]$.)

4 The *market-game* with contingent offers

It is well-known that the market-game with non-contingent offers does not allow much feedback from the actions of others to that of any one agent. That is why most of the literature on information-aggregation and on strategic foundations for CE uses the market game with limit orders or supply functions—which, of course, are versions of contingent offers. In the setting we are studying, it is natural to use as contingencies the four-element support of (γ, N) so that each young person submits a four-tuple of saving amounts.

Before receiving the offers, the market-game mechanism knows only the pre-transfer quantity of money; it does not know the current realization of (γ, N). If the old offer all their money, then after it receives the offers of the old, it knows γ; and if all the young submit offers, then after it receives the offers of the young, it knows N. Hence, it can use the contingent saving offers that correspond to the actual realization of (γ, N).[6]

Once the mechanism has determined the realizations of (γ, N), it determines the outcomes in the same way as in the market-game with non-contingent offers. An equilibrium is defined the same way as before, but the interpretation of x, x_- and x_+ is different. Now, regardless of what we assume about what the young know, they are all mappings with the four-element support of (γ, N) as their domain.

Our claim (see the second row of Table 1) is that independent of what the young know about the current realization of (γ, N), the only active-trade stationary limiting allocation from among the C, L, and N allocations is the N-allocation.

There are two things to prove.

Proposition 2. *Fix what the young know. Neither the* C*-allocation nor the* L*-allocation is a limiting equilibrium of the market game with contingent offers.*

Proof. Consider first the limiting C-allocation. Suppose that $x_- = x_+ = \hat{x}$ is that allocation. Then, the implied return distribution is that in (9). Facing that return distribution, a defector can submit a saving offer that depends on N. The defector wants to do that unless optimal saving when the return distribution is uniform over $\{\frac{N_l}{N_l}, \frac{N_h}{N_l}\}$ is the same as that when the return distribution is uniform over $\{\frac{N_l}{N_h}, \frac{N_h}{N_h}\}$. Because the former return distribution dominates the latter, the gross-substitutes

[6] To have a complete description of the mechanism, we should also describe how the mechanism selects among contingencies if not all the money is offered and if some of the young do not submit offers. We can have the mechanism choose the contingency $\gamma = \gamma_l$ if the total offer of money satisfies $M \leq (1 + \gamma_l)M_-$, where M_- is the known previous quantity of money, and choose the contingency $\gamma = \gamma_h$ otherwise. Similarly, we can have the mechanism choose the contingency $N = N_l$ if the total number of offers submitted by the young does not exceed N_l and have it choose $N = N_h$ otherwise.

assumption implies that the defector wants to choose different contingent offers, offers that are contingent on the current N.

Next consider the limiting L-allocation. Suppose that $x_- = x_+ = \hat{x}$ is that allocation, and has $\hat{x}_m$ offered when $(\gamma, N) = (\gamma_l, N_l)$ and when $(\gamma, N) = (\gamma_h, N_h)$. Then a potential defector who chooses one offer for $(\gamma, N) = (\gamma_h, N_h)$ and a different one for $(\gamma, N) = (\gamma_l, N_l)$ can choose distinct offers for those two realizations; one that is optimal for the return distribution in (10) and another that is optimal for the return distribution in (12). Again, the former return distribution dominates the latter so that the gross-substitutes assumption implies that the defector wants to choose different contingent offers for those two contingencies. □

Proposition 3. *Fix the information of the young and fix k. The unique active-trade stationary equilibrium is the same as under non-contingent offers when the young know N.*

This conclusion follows directly from the proof of Proposition 1.

5 Closing Remarks

We have taken it for granted that it is desirable to analyze models with incomplete information strategically. That is not a new point of view (see, for example Dubey *et al.*, 1987). When we do so using our two versions of the market-game, the allocation that gives rise to a positive correlation between total output and the growth rate of the stock of money is questionable on two grounds. Under the informational assumption that the young do not see contemporaneous realizations of the shocks, neither of the two strategic formulations we have studied gives rise to that allocation as a limiting equilibrium. That allocation *is* a limiting equilibrium in the version with non-contingent offers if the young see the ratio of the shocks, but viewed as a signal the ratio is a very special signal.

Of course, our results do not rule out the possibility that some other strategic formulation would select the allocation that gives rises to a positive correlation. However, in order for the counter-example to be convincing, neither the environment nor the game that agents play should be too strange because a counterexample is meant to be descriptive or an instance of *positive* economics. The environment is not strange although some have quarrelled with it.[7] As regards the mechanism for accomplishing trade, it is hard to imagine anything simpler than the market-game with non-contingent offers.

We do not view our results as suggesting that incomplete-information models cannot give rise to positive correlations between total output and the growth rate of the stock of money. One direction to pursue is models of decentralized trade as in Araujo and Shevchenko (2006). In that model trade occurs in pairs and the important source of incomplete information is that one pair does not see what is happening contemporaneously among other pairs. And, of course, there is a large literature on "sticky-information" models.

Appendix A Proof of Proposition.

A.1 Notation.

To be able to express the proof for each information structure in a single framework, we introduce some notation. Let $\mathcal{I} = \{N_l, N_h\} \times \{\gamma_l, \gamma_h\}$ denote the set of possible states and corresponding to each information structure j, consider the partition $\mathcal{I}_j$ that groups all indistinguishable states:

$$
\begin{aligned}
\mathcal{I}_i &= \{\mathcal{I}\} \\
\mathcal{I}_{ii} &= \Big\{\big\{(N_l, \gamma_h)\big\}, \big\{(N_l, \gamma_l), (N_h, \gamma_h)\big\}, \big\{(N_h, \gamma_l)\big\}\Big\} \\
\mathcal{I}_{iii} &= \Big\{\{s\} | s \in \mathcal{I}\Big\}
\end{aligned}
$$

[7] Barro (1989, pages 2 and 3) suggests that the model prejudices the result because transfers go entirely to buyers (the old), as opposed to producers (the young). And others have questioned the relevance of assuming a lag in observing the quantity of money.

Moreover, for each information structure j, let $\pi_j^s : \{N_l, N_h\} \to [0,1]$ describe the belief of the young generation about its size at state s:

$$\pi_i^s(N) = 1/2 \quad \text{for all } s, N$$

$$1 - \pi_{ii}^s(N_h) = \pi_{ii}^s(N_l) = \begin{cases} 1 & \text{if } s = (N_l, \gamma_h) \\ 0 & \text{if } s = (N_h, \gamma_l) \\ 1/2 & \text{otherwise} \end{cases}$$

$$\pi_{iii}^s(N) = \begin{cases} 1 & \text{if } s = (N, \gamma) \text{ for some } \gamma \\ 0 & \text{otherwise} \end{cases}$$

Given a replication factor k, an information structure j, the average real savings of the other members of the current cohort $(x_-^s)_{s\in\mathcal{I}} \in (0,\omega]^4$, and the average real savings of the members of the next cohort $(x_+^s)_{s\in\mathcal{I}} \in (0,\omega]^4$, the expected utility of saving x in state s is given by

$$P_k^j(x; s, x_-, x_+) = u(\omega - x) + \sum_{N,N',\gamma'} \frac{1}{4}\pi_j^s(N) v\Big(R_k\Big(x, x_-^s, x_+^{(N',\gamma')}\Big)x\Big)$$

$$\text{where} \quad R_k(x, x_-^s, x_+^{s'}) = \frac{kN' x_+^{s'}}{(kN-1)x_-^s + x}$$

Finally, we find it helpful to define two functions $f : (0,\omega) \to \mathbb{R}_{++}$ and $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ as follows:

$$f(x) = xu'(\omega - x)$$
$$g(x) = xv'(x)$$

Now, we can rewrite our definition of equilibrium as follows:

Definition. *Given a replication factor k and a specification of the information structure j, a symmetric, stationary, active-trade Nash equilibrium is an indexed collection $(\hat{x}^s)_{s\in\mathcal{I}} \in (0,\omega]^4$ that satisfies two properties:*

(a) *$\hat{x}$ is constant over each element of the partition $\mathcal{I}_j$, and*

(b) $\hat{x}^s = \arg\max_x P_k^j(x; s, \hat{x}, \hat{x})$

A.2 Proof.

The proof of our main result consists of a series of lemmas. We first argue that $P_k^j(\cdot; s, x_-, x_+)$ is concave for all s and positive x_-, x_+. This allows us to define a system of equations that characterizes symmetric, stationary, active-trade Nash equilibria for any replication factor k. To show that the system of equations has a positive solution, we apply a transformation to the system of equations to define a mapping whose fixed points can be traced back to equilibria. The way the mapping is defined and the proof that it has a unique fixed point follows closely the idea used in Lucas (1972). Moreover, we argue that the function mapping the replication factor to fixed points is continuous in k. This implies the last part of the proposition.

Lemma 1. *Given a replication factor k and information structure j, $P_k^j(\cdot; s, x_-, x_+)$ is concave for all s and positive x_-, x_+.*

Proof. Note that $x \mapsto R_k(x, x_-, x_+)x$ is concave for all positive x_-, x_+. One way to see this is by noting that its first derivative is decreasing:

$$\frac{\partial R_k(x, x_-, x_+)x}{\partial x} = \frac{kN'(kN-1)x_+x_-}{((kN-1)x_- + x)^2}$$

Because v is increasing, the composition of v with $x \mapsto R_k(x, x_-, x_+)x$ is also concave. Hence, the function of interest is concave because it is a linear combination of concave functions with nonnegative coefficients. □

The concavity of $P_k^j(\cdot; s, x_+, x_-)$ and $u'(0) = v'(0) = \infty$ imply that the first order optimality condition is necessary and sufficient. Moreover, our notion of equilibrium requires the optimal savings of all people in all generations to be equal. This leads to the following lemma.

Lemma 2. *Given a replication factor k and information structure j, $x = (x^s)_{s \in \mathcal{I}}$ is a symmetric, stationary, active-trade Nash equilibrium if and only if x is the solution to the following system of equations:*

$$f(x^s) = \sum_{N, N', \gamma'} \frac{1}{4} \pi_j^s(N) \left(1 - \frac{1}{kN}\right) g\left(\frac{N'}{N} x^{(N', \gamma')}\right) \quad \text{for all } s \tag{17}$$

Proof. Taking the derivative of P_k^j with respect to x and then equating $x_- = x_+ = x$ gives us (17). This implies that x satisfies part (b) of the definition of equilibrium.

To see that it also satisfies part (a), note that f is increasing and the right-hand sides of (17) are equal for any two states that belong to the same element of $\mathcal{I}_j$. $\square$

In what follows, any mention of j refers to a fixed information structure $j \in \{\mathcal{I}_i, \mathcal{I}_{ii}, \mathcal{I}_{iii}\}$. The value of j does not affect the correctness of the statements.

Now, note that given our assumptions on u, the inverse of f, $f^{-1}: \mathbb{R}_{++} \to (0, \omega)$ exists and is increasing. Define the mapping $T_\kappa : \mathbb{R}^4 \to \mathbb{R}^4$ as $T_\kappa(y) = (t_s(y))_{s\in\mathcal{I}}$ where

$$t_s(y) = \ln \sum_{N,N',\gamma'} \frac{1}{4}\pi_j^s(N)\left(1 - \frac{\kappa}{N}\right) g\left(\frac{N'}{N} f^{-1}(e^{y^{(N',\gamma')}})\right)$$

Then $(y^s)_{s\in\mathcal{I}}$ is a fixed point of T_κ if and only if $(f^{-1}(e^{y^s}))_{s\in\mathcal{I}}$ solves (17) for replication factor $k = 1/\kappa$.

Lemma 3. *T_κ is a contraction uniformly over $\kappa \in [0, 1]$ with respect to the uniform norm: for any $y, z \in \mathbb{R}^4$ and $\kappa \in [0, 1]$*

$$\|T_\kappa(y) - T_\kappa(z)\|_\infty \leq (1 - a)\|y - z\|_\infty$$

Proof. First, note that by (3) we have

$$\frac{xg'(x)}{g(x)} = 1 + \frac{xv''(x)}{v'(x)} \in (0, 1 - a] \tag{18}$$

Moreover,

$$\frac{x(f^{-1})'(x)}{f^{-1}(x)} = \frac{x}{(f \circ f^{-1})(x) - (f^{-1}(x))^2 u''(\omega - f^{-1}(x))} \in (0, 1) \tag{19}$$

Next, consider

$$\frac{\partial}{\partial x} \ln g\left(\frac{N'}{N} f(e^x)\right) = \left(\frac{g'\big(N'/Nf(e^x)\big)N'/Nf(e^x)}{g\big(N'/Nf(e^x)\big)}\right)\left(\frac{e^x f'(e^x)}{f(x)}\right)$$

By (18) and (19), these factors lie in $(0, 1 - a]$ and $(0, 1)$, respectively. Hence, for any N and N'

$$0 < \frac{\partial}{\partial x} \ln g\left(\frac{N'}{N} f(e^x)\right) \leq 1 - a \tag{20}$$

Now, given s

$$|t_s(y) - t_s(z)| = \left| \ln \left(\sum_{N,N',\gamma'} \alpha(N,N',\gamma') \frac{g\left(N'/N f^{-1}\left(e^{y^{(N',\gamma')}}\right)\right)}{g\left(N'/N f^{-1}\left(e^{z^{(N',\gamma')}}\right)\right)} \right) \right|$$

where

$$\alpha(N,N',\gamma') = \frac{\pi_j^s(N)\,(1-\kappa/N)\, g\left(N'/N f^{-1}\left(e^{z^{(N',\gamma')}}\right)\right)}{\sum\limits_{\hat{N},\hat{N}',\hat{\gamma}'} \pi_j^s(\hat{N}) \left(1-\kappa/\hat{N}\right) g\left(\hat{N}'/\hat{N} f^{-1}\left(e^{z^{(\hat{N}',\hat{\gamma}')}}\right)\right)}$$

Note that $\alpha > 0$ and $\sum\limits_{N,N',\gamma'} \alpha(N,N',\gamma') = 1$. It follows then from the quasiconvexity of $x \mapsto |\ln(x)|$ that

$$|t_s(y) - t_s(z)| \leq \max_{s\in\mathcal{I}} \left| \ln g\left(\frac{N'}{N} f^{-1}(e^{y^s})\right) - \ln g\left(\frac{N'}{N} f^{-1}(e^{z^s})\right) \right|$$

Application of the mean value theorem and (20) gives

$$|t_s(y) - t_s(z)| \leq \max_{s\in\mathcal{I}} |(1-a)(y^s - z^s)| = (1-a)\,\|y-z\|_\infty$$

Thus,

$$\|T_\kappa(y) - T_\kappa(z)\|_\infty \leq (1-a)\|y-z\|_\infty$$

□

Lemma 4. *For each replication factor k there exists a unique solution $\hat{x}_k$ to* (17)*. Moreover, $\hat{x}_k$ converges to $\hat{x}$ where $\hat{x}$ is a solution to*

$$f(x^s) = \sum_{N,N',\gamma'} \frac{1}{4}\pi_j^s(N) g\left(\frac{N'}{N} x^{(N',\gamma')}\right) \quad \text{for all } s$$

Proof. Given $\kappa \in [0,1]$, T_κ is a contraction that maps $\mathbb{R}^4$ to itself. By the contraction mapping theorem, T_κ has a unique fixed point y_κ for each κ in the unit interval. It follows that for each $k \in \mathbb{N}$, $\hat{x}_k = \left(f^{-1}(e^{y^s_{1/\kappa}})\right)_{s\in\mathcal{I}} > 0$ is a solution to (17).

Furthermore, since T_κ is a contraction uniformly over κ and $T_\kappa(y)$ is continuous in κ for each fixed y, $\kappa \mapsto y_\kappa$ is continuous (see, for example Krantz and Parks, 2012, Proposition 3.4.5). Hence $y_{1/k} \to y_0$ as $k \to \infty$. Moreover, because $y \mapsto f^{-1}(e^y)$ is continuous over $\mathbb{R}_{++}$, it follows that $\hat{x}_k \to f^{-1}(e^{y_0}) > 0$ as $k \to \infty$. □

It should be mentioned that given the finiteness of the support of the shocks in our model, the strengthening of concavity required by assumption (3) is not necessary. If we were to drop the lower bound on $-v''(z)z/v'(z)$, we could alternatively define the mapping T over a compact set of the form $[\ell, \upsilon]^4$ where $t_s(\ell) > \ell$ and $t_s(\upsilon) < \upsilon$ for all s. To accomplish this, ℓ can be chosen using a technique similar to that in Wallace (1992) and Manuelli (1986), with some adjustments to incorporate the transformation applied to the original system of equations. A similar logic can be used to find an appropriate υ. We have chosen not to pursue this strategy in the interest of brevity.

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