DOCUMENTOS DE TRABAJO

Comparison of Model for Pricing Volatility Swaps

Néstor Romero

N.º 708 Diciembre 2013

BANCO CENTRAL DE CHILE





DOCUMENTOS DE TRABAJO

Comparison of Model for Pricing Volatility Swaps

Néstor Romero

N.º 708 Diciembre 2013

BANCO CENTRAL DE CHILE





CENTRAL BANK OF CHILE

La serie Documentos de Trabajo es una publicación del Banco Central de Chile que divulga los trabajos de investigación económica realizados por profesionales de esta institución o encargados por ella a terceros. El objetivo de la serie es aportar al debate temas relevantes y presentar nuevos enfoques en el análisis de los mismos. La difusión de los Documentos de Trabajo sólo intenta facilitar el intercambio de ideas y dar a conocer investigaciones, con carácter preliminar, para su discusión y comentarios.

La publicación de los Documentos de Trabajo no está sujeta a la aprobación previa de los miembros del Consejo del Banco Central de Chile. Tanto el contenido de los Documentos de Trabajo como también los análisis y conclusiones que de ellos se deriven, son de exclusiva responsabilidad de su o sus autores y no reflejan necesariamente la opinión del Banco Central de Chile o de sus Consejeros.

The Working Papers series of the Central Bank of Chile disseminates economic research conducted by Central Bank staff or third parties under the sponsorship of the Bank. The purpose of the series is to contribute to the discussion of relevant issues and develop new analytical or empirical approaches in their analyses. The only aim of the Working Papers is to disseminate preliminary research for its discussion and comments.

Publication of Working Papers is not subject to previous approval by the members of the Board of the Central Bank. The views and conclusions presented in the papers are exclusively those of the author(s) and do not necessarily reflect the position of the Central Bank of Chile or of the Board members.

Documentos de Trabajo del Banco Central de Chile Working Papers of the Central Bank of Chile Agustinas 1180, Santiago, Chile Teléfono: (56-2) 3882475; Fax: (56-2) 3882231

COMPARISON OF MODEL FOR PRICING VOLATILITY SWAPS*

Néstor Romero Gerencia de Investigación Financiera Banco Central de Chile

Abstract

The popularity of volatility derivatives has increased through these years of financial turmoil. In particular, variance and volatility swap seem interesting to analyse due to its growing trading volume. Hence, the aim of this work is to present a full revision of these two volatility derivatives, comparing pricing methodologies, like Taylor expansion and Heston (1993) volatility process. In addition, there will be a complete section dedicated to the study of the volatility skew and the wings or "smile". The results showed that the Taylor expansion has a reasonable level of convergence at some values of the parameters of the volatility dynamics, though the findings concluded that higher order of this expansion yielded poor results than the lower order. In the other hand, the smile and the volatility skew showed that the former may change the final value of the fair volatility strike, whereas the latter has almost null impact on this one.

Resumen

La popularidad de instrumentos de volatilidad, como los swaps de volatilidad y varianza, se ha incrementado incluso durante los años de crisis financiera. El presente trabajo busca exhibir una revisión completa de estos dos derivados, comparando metodologías como la expansión de Taylor y el proceso de volatilidad descrito por Heston (1993), además de dedicar un apartado completo al estudio del *skew* y *smile* de la volatilidad. Los resultados muestran que la expansión de Taylor de segundo orden, a ciertos parámetros, tiene un nivel de convergencia razonable al descrito por Heston, sin embargo esto no se observa al tratar con una expansión de orden mayor, como la de tercer orden. Por otro lado, la *smile* y la *skew* muestran que cambios en la última producen cambios en el valor justo del *strike price* del swap de volatilidad, mientras que la primera no produce grandes cambios en éste. Lo anterior es un resultado a tener en cuenta para quienes realizan el *pricing* de estos derivados, ya que cambios en la *skew* producirían cambios en el valor final del swap.

^{*}Email: <u>nromero@bcentral.cl</u>.

1 Introduction

It has been nearly 20 years since the first attempts to obtain pure exposed volatility assets quoted in the market. About in 1993 the first volatility product appeared in the market according to Carr and Lee (2009)¹, and it was a variance swap dealt by an investor and the Union Bank of Switzerland (UBS). Since that event, volatility products tended to become more popular, though between 1993 and 1998 there were not traded very often, according with the same authors. They also mentioned in their work that it was not after 1998 that variance swaps were already established as trading products in OTC markets, since its hedging process was relatively easy to carry out in comparison with the volatility swaps whose popularity was further below.

In this present work, the aim in general terms is to provide with an exhaustive analysis of the pricing methodology of volatility and variance swaps. In particular, the main focus will be in the pricing method of the fair variance and volatility strike price. These two prices represent a key feature in the payoff function of both swaps, for which most of the literature revised has focused on providing methodologies to price them.

One may wonder why to conduct this research only in these two assets. Well, the answer might be that since the OTC market for these two products have grown further from the early years of the past decade, especially in Asian markets, the need of new techniques to price them have also grown. In this line, researchers have proposed different pricing methods, in which one might find from model independent methods to approaches based in volatility dynamics process. Thus, the purpose in this research will be to mention some of these approaches and apply them to observe the differences on the values of the products. Regarding the approaches to price the fair variance and volatility strike prices, one will be based on a Taylor expansion with different orders, in which the first one will be model independent and the higher order will be based on Heston volatility dynamics. Another approach described on this work is to observe how the price of the fair volatility strike is affected by changes in an implied variance smile and its volatility skew. The latter methodology is a combinations of two different approaches in which one is a parameterization of an implied variance function and the other one is replication function based on a static option portfolio whose volatility value will be obtained from the parameterization. A further explanation of each of these methodologies mentioned will be provided in the following sections. In addition, a more intuitive analysis will be also provided with respect to the benefits and drawbacks of these two volatility derivatives and also a description of their payoff functions may be found on the methodology section.

Thus, the structure of this work is as follows. In the second section, the literature about pricing volatility and variance swaps will be listed. In the third section, the methodology selected to price these two assets will be described in deep. Afterwards, the results will be presented and analysed in the fourth section. In the fifth section, limitations of the currently used methodology will be explained along with possible future research. Finally, all the findings will be summarised at the end of this research.

¹ Other works in which these securities are revised are Brockhaus and Long (2000), Neuberger (1994), Carr and Madan (1997), Dupire (1993) and Carr and Lee (2007) among others.

2 Literature Review

2.1 Obtaining Volatility Exposure Assets

Carr and Madan (1997) represent one of the first attempts in deriving volatility payoff functions by using pure volatility exposure instruments. In their work they began by mentioning some methods that allows the investor to obtain some exposure to realised volatility. These methods were first to take a static position in options, the second method is related to delta hedging option positions and the third method is to take position in an over the counter asset whose payoff could be expressed in terms of an implicit function of volatility. Thus, the focus of this work will be in the first method, the static position in options whose simplest example may come from taking a position in a straddle strategy. The theory behind this strategy is that one can use at the money option to forecast the realised volatility and therefore gain exposure to volatility. In particular, there is a replication strategy mentioned by the authors, in which one can replicate the terminal price of an asset by using a decomposition formulae based on static positions taken in options at time equal zero. This formula is presented by the following equation.

$$f(F_T) = f(\kappa) + f'(\kappa)[(F_T - \kappa)^+ - (\kappa - F_T)^+] + \int_0^{\kappa} f''(K)(K - F_T)^+ dK + \int_{\kappa}^{\infty} f''(K)(F_T - K)^+ dK$$
(1)

Equation (1), represents a useful technique to attempt to price volatility swaps based upon options on realised variance. In this approach, the authors did not assume to have constant volatility as the options theory did, and also, this method is described to be model free. At the end, their approach is proved to be sensitive to volatility movements, but it has the drawback that one may obtain considerable exposure in terms of the price of the underlying if this one moves from its central value.

Another work that is worth to mention with respect to volatility assets is the one written by Carr and Lee (2009). They have surveyed a rather substantial number of past papers regarding the market for volatility derivatives and instruments like volatility and variance swaps. In addition, they explained that the volatility measure that is utilised to describe the payoff of any volatility asset may be either the implied volatility obtained from option prices or the realised volatility described from prices of some underlying asset. Moreover, the authors have mentioned one problem and one remark related to the measurement of volatility assets. Regarding the problem, this one is how to reduce the error or the difficulties of the hedging process, for which they have proposed a different robust replication strategy for variance swaps that are monitored discretely in order to solve it. With respect to the remark, the authors have proposed to continue with the exploration of the connexion between volatility derivatives and the implied volatility of vanilla options in models that should be more realistic.

2.2 Trading Volatility and Variance Swaps

The aim of this sub section is to improve the intuition of the trading process of these two assets. Thus, one might wonder when is the appropriate time to take a position in volatility swaps or in variance swaps. This concern is addressed by the works of Pengelly (2009) and Clark (2010).

According to Clark (2010), it can be suggested in terms of variance and volatility swaps that when financial distress approaches, one possible strategy to follow is to move from the first to the second instrument mentioned. On the contrary, in markets with a certain level of stability, traders may change their scheme to take positions in variance swaps again. The latter strategy provided seems to be rather

simple and easy to implement. However, this approach lacks of a further analysis in terms what underlying asset should be used, and even more, it does not mention what benefits and drawbacks the variance and volatility swaps may have. Thus, as both works mentioned, with respect to the underlying asset it is known that a single stock can be more volatility than an index, and then a trader should adapt his strategy according to the features of the underlying asset. For instance, given that single stock movements often are more volatile, investors preferred to take long positions in variance swaps whose underlying are a single stock to take advantage of the convexity² of this instrument given the volatility of this one. In addition, variance swaps can be utilised as an instrument to gain correlation to some product in which traders might be short of. Despite of these good characteristics of the variance swaps, the problems arise when the volatility increases in the market and the investors have taken short position on variance swaps, which is the scenario described by Pengelly (2009). The sellers of variance swaps have to pay the floating part of the deal, hence when there is a sudden increase of volatility their losses will likely to be higher than their profits.

In terms of trading, in Goldman Sachs (1996), there is a description of the agents in terms of the usage of volatility instruments. Therefore, the usages that one can find are: "Directional Trading of Volatility Levels", "Trading the Spread between Realized and Implied Volatility" and "Hedging Implicit Volatility Exposure". The first usage is related to speculators gaining more exposure to volatility than the exposure obtained by trading options. Regarding the second usage, investors trade on the difference between the realised and implied volatility in which the key of this strategy is that the investor has to leave its position taken in the swap before maturity so he may be able to trade the spread. In terms of the last usage, there are risk arbitrageurs and hedge funds that may take a long position in shares of a company that is merger to another. If the conditions of the market turn out to be not competitive as expected then the merger is likely not to succeed due to an increase of the volatility. The latter example shows an implicit exposure to overall market volatility that one may want to hedge.

2.3 Hedging Variance and Volatility Swaps

As part of the intuition of the trading process, one key factor that must be analysed is the hedging process. This is an issue addressed in a publication released by Goldman Sachs (1999) in which they mention that variance swaps may be easily hedged by selecting a right vanilla options portfolio whose strike prices need to be carefully chosen. Furthermore, in order to maintain the hedge traders constantly have to rebalance the position taken by performing a dynamic hedging. This strategy might be quite expensive and also requires of almost an endless supply of options, a weakness of the variance swap hedging since it might be very difficult to have a smooth supply of options. Regarding the volatility swap hedging, to take a hedging strategy based on standard options might not be a good approach as such options may lead the position to be exposure to other factors rather than just volatility. Thus, one strategy that might be successful, according to the publication mentioned before, in hedging volatility swaps is to hedge it by performing a dynamic hedging using variance swaps.

In terms of hedging volatility, Neuberger (1994) has published one of the first works regarding volatility hedging instrument. Given that traders were using delta hedging strategy, they were able to eliminate the risk of asset's movements but, as it has been said before, there were not able to eliminate the volatility risk. Thus, the author tried to respond to this matter by introducing a new instrument which he has called the Log Contract. This new instrument is a future style contract in which its settlement price is equal to the logarithm of the underlying asset price. In particular, one of the features of this new instrument is to ensure a payoff that depends only on the difference between the

² This feature is explained in the following section.

volatility expected at initial time of the contract and the actual volatility that occurs over the life of this one. Another quality of this contract is that maintain its sensitivity to volatility no matter the asset price.

2.4 Description of Variance and Volatility Swaps

In terms of a precise definition of these instruments one may refer to Broadie and Jain (2008) as a rather excellent source. Since these two instruments are swaps, their structure is similar to vanilla interest rate swaps, in which there is a floating leg and a fixed leg, and also there is a notional amount that multiplied the difference of the first two terms. According to Broadie and Jain (2008) definitions, one may describe the variance swaps as a contract over a realised variance and a fixed price, accorded by the parties, times a notional amount. With respect to the realised variance, this one is calculated from a given asset which might be a single stock or an index, depending of the level of volatility that the investor wants to be exposed. Thus, it is possible to state that the variance swap payoff is a linear function of the realised variance. In the other hand, the volatility swap represents a contract over a realised volatility and a strike price times a notional amount whose payoff is a concave function of the realised variance or a linear function of the realised variance of the underlying asset selected. As it might be known, variance and volatility are not traded assets as equity stocks are, and both need to be calculated as showed in an article of BNP Paribas Bank (2005).

$$\sigma^{2} = \frac{252}{T} \sum_{i=1}^{T} \left(\ln\left(\frac{S_{i}}{S_{i-1}}\right) \right)^{2}$$
(2)
$$\sigma = \sqrt{\frac{252}{T}} \sum_{i=1}^{T} \left(\ln\left(\frac{S_{i}}{S_{i-1}}\right) \right)^{2}$$
(3)

From equations (2) and (3) one may observe how the variance and volatility are calculated, in discrete time, in terms of the log daily returns of a stock or an index, and also in terms of trading day instead of calendar days.

2.5 Structure and Pricing process of Variance and Volatility Swaps

Brockhaus and Long (2000) approaches will be the ones followed in this work, which means the volatility dynamics selected will be Heston model defined on Heston (1993). This approach is a stochastic volatility model in which the structure was created in order to have correlation arbitrarily between the asset return and its volatility. The underlying asset is defined by a geometric Brownian motion and its variance is defined as a square process.

$$dS(t) = \mu S dt + \sqrt{\nu(t)} S dZ_1(t)$$
(4)

$$d\sigma^2 = \kappa (\theta^2 - \sigma_t^2) dt + \gamma \sigma_t dZ_t^2$$
(5)

Equations (4) and (5) describe the processes mentioned before, in which equation (5) is presented as a mean reverting process, meaning that the variance will tend to converge towards its mean value. Regarding Heston's work, a deeper description of it is beyond the scope of this research. A final remark is that this model works under risk neutrality which means that the probabilities are risk neutral, hence the scenario described under Heston model will be arbitrage free.

A more intuitive description of Heston model may be found in Gatheral (2006) in which the author explains the process that is described by Heston and also provides with a further analysis in terms of

its usage and its popularity. For example, Gatheral mentions that Heston (1993) showed the first approach of a mean reverting process, which might be called a Cox, Ingersoll and Ross or CIR (1985) process, applied to describe volatility movements.

Regarding the pricing approach of variance and volatility swaps, Brockhaus and Long (2000) have listed in his research some methods to price them by calculating the fair variance and volatility strike price of the payoff function. The authors first described that there are two types of volatility models, the deterministic and the stochastic ones. In the deterministic models, the volatility depends on the level of the underlying asset whereas in the stochastic models the volatility follows a stochastic process in which Brockhaus and Long (2000) refers to CIR (1985) or mean reverting process³. In addition the author defines the realised variance as the average of a continuous variance over time, which will be used as the floating part of variance and volatility swaps. Thus in order to calculate the fixed part of the swap, the fair variance and volatility strike price, the authors detailed an approach to calculate them based on a Taylor expansion. The Taylor expansion is carried out in terms of the square root of the variance function, around a certain volatility value. This expansion is on terms of the expected variance and variance of the variance, in which both are calculated from Heston model. To provide with a further analysis, the researchers also calculated an exact solution for the fair volatility strike price obtained from applying numerical methods to the Laplace transformation of the variance function in which the expected value and variance of the variance taken from Heston are also utilised in the calculation.

In terms of the intuition, Brockhaus and Long (2000) refers to the volatility swaps as an asset more difficult to price than the variance swap but also more desired by investors. One of the problem that make volatility swaps difficult to price is the hedge part of the whole process. It is well known that if one price a new asset also needs to hedge this asset in order to cover the position taken. Furthermore, a far important reason relies on the arbitrary condition in which an asset should be possible to hedge, otherwise the condition of arbitrage free will not be satisfied.

Another approach to price volatility swaps come from Carr and Madan (1997), which has been already mentioned in equation (1). In particular, for pricing the fair volatility strike, this method requires of options on realised variance. These options are hard to be found in any market data base, so they will be constructed based on a parameterization described in Gatheral (2006). The first aim of this parameterization is to obtain an implied variance from a series of κ values and other parameters. In this method κ is defined as

$$\kappa = Log\left(\frac{K}{f}\right)$$

And the parameterization is presented below.

$$Var(\kappa, a, b, \sigma, \rho, m) = a + b\left\{\rho(\kappa - m) + \sqrt{(\kappa - m)^2 + \sigma^2}\right\}$$
(6)

With respect to the mechanism of equation (6), as one changes all these parameters, one may observe changes on the implied variance smile function. Once that the implied variance is obtained, one may calculate options on realised variance, and then these ones will be included as inputs on equation (1). Finally, the ultimate aim of this parameterization is to observe how the changes in the smile of the implied variance and the volatility skew may change the final outcome of the fair volatility strike price.

³ In particular, Heston volatility dynamics is the process selected, as it has been mentioned early.

The above methodology of pricing volatility swaps is similar to the one applied in Carr and Lee (2007), in which the authors show how to price volatility and variance options via swaps on those two assets. One of the cases that these two researchers had to deal with is the case in that variance and volatility swaps are not available to trade. In such a case, they have proposed a methodology to synthesise them by using vanilla options, similar to the approach proposed by Carr and Madan (1997). However, in this case, Carr and Lee (2007) have taken a static position in options and a dynamic position in shares as well. As result, they have created a variance and volatility swaps or, as they have called them, a "Synthetic Volatility and Variance Swaps". Thus the purpose of mention Carr and Lee (2007) approach is that this serves as evidence to support the methodology described by the combination of Gatheral's parameterization and Carr and Madan replication approach.

3 Methodology

3.1 A Description of Volatility Swaps and Variance Swaps Payoff Functions

Before revising the pricing methodology, it might be helpful to state a description of the variance and volatility swaps, provided by Broadie and Jain (2008), in terms of their payoff function in a deterministic world.

$$(V_d(0,n,T) - K)N \tag{7}$$

According to the definition, the variance swap payoff function stated in equation (7) is linear in realised variance. In this case, equation (7) represents the realised variance over stock returns in which the life of the contract begins at time 0 and ends at time T, with increments of n. The variance strike price is defined by K, which is accorded by both parties entering in the contract, and N represents the notional amount. Thus, if one is entering in a payer variance swap contract it will pay a fix amount defined by the variance strike price and it will receive the floating leg defined by the realised variance, times the notional amount. In the contrary, the receiver variance swap will receive the fixed leg and it will pay the realised variance. With respect to the volatility swap payoff function, this one is described in the following equation.

$$\left(\sqrt{V_d(0,n,T)} - K\right)N\tag{8}$$

Equation (8) shows that in this case the payoff of the volatility swap is a square root function of the realised variance. In other words, the volatility swap payoff is a linear function on realised volatility, since this one is a square root function of the realised variance. The procedure is likely to be the same as described above, in terms of the payer and receiver swap.

Now, the same description will be carried out by using a stochastic variance model. This model is obtained from the findings in Heston (1993), and is defined by equations (4) and (5) in the above section. Thus, Broadie and Jain (2008) stated that the continuous realised variance function is the one described in equation (9).

$$V_c(0,T) = \frac{1}{T} \int_0^T v_s ds \tag{9}$$

In addition, in order to build a payoff function, the fair variance strike prices is defined, according to Broadie and Jain (2008), as the one that makes the net present value of the payoff function equal to zero. The equation that defines the contract in this framework is the following one.

$$E_0^Q[e^{-rT}(V_c(0,T) - K_{var}^*)] = 0$$
(10)

From equation (10), in which E_0^Q denotes expectation at time zero under risk neutral measure, is possible to observe the fair variance strike price or K_{var}^* , which is defined by the following equation.

$$K_{var}^* = E[V_c(0,T)] = E\left(\frac{1}{T}\int_0^T v_s ds\right) = \theta + \frac{v_0 - \theta}{\kappa T}(1 - e^{-\kappa T})$$
(11)

Hence, equation (11) represents the fair variance strike price that makes equation (10) equal to zero. It is worth to mention that this definition of the fair variance strike price is based in Heston dynamics parameters and the procedure stated by the authors is similar to the one followed by Brockhaus and Long (2000). In addition, Broadie and Jain (2008) have also provided with a contract for the volatility swap based in the stochastic volatility model.

$$E_0^Q \left[e^{-rT} \left(\sqrt{V_c(0,T)} - K_{vol}^* \right) \right] = 0$$
 (12)

From equation (12) one might observe that the fair volatility strike price or K_{vol}^* is also the one that makes the equation (12) equal to zero.

$$K_{vol}^* = E\left(\sqrt{\frac{1}{T}\int_0^T v_s ds}\right) = E\left[\sqrt{V_c(0,T)}\right]$$
(13)

From equation (13) one might observe that the fair volatility strike price is just the expected value of the square root function of the continuous realised variance. Now, there is a remark related to the fair variance and volatility strike price that is worth to state, which is the following one.

$$K_{vol}^{*} = E\left[\sqrt{V_{c}(0,T)}\right] \le \sqrt{E[V_{c}(0,T)]} = \sqrt{K_{var}^{*}}$$
(14)

In equation (14) one might see that the fair volatility strike price is bounded from above by the square root of the fair variance strike price. This result is possible to obtain by applying Jensen's Inequality, and it is referred in the literature review as the "volatility convexity" or as the "convexity correction" as mentioned in Broadie and Jain (2008).

3.2 Taylor Expansion Method

As Broadie and Jain (2008) have described, in order to calculate the value of the payoff function of a variance and volatility swap, one needs to calculate the fair variance and volatility strike price. Thus, a Taylor expansion will be derived over the square root of the variance function around some single point which should be very close to the expected value of the fair volatility strike price. This methodology has been proposed by Brockhaus and Long (2000).

First of all, it is necessary to state the function to be used and obtain its first and second order approximation as it was showed in Brockhaus and Long (2000). This function is the square root of the variance and is presented below in equation (15).

$$F = \sqrt{V} \tag{15}$$

The fair volatility strike price will be obtained from taking the expectation of equation (15). In addition, as the derivation of the Taylor expansion of equation (15) reaches higher values, the interaction of the fair volatility and variance strike price will be possible to observe. Thus, the first order approximation is the following.

$$\sqrt{V} \approx \sqrt{V_0} + \frac{(V - V_0)}{2\sqrt{V_0}}$$
(16)
$$= \frac{(V + V_0)}{2\sqrt{V_0}}$$

Equation (16) represents the first order approximation of the square root function showed in equation (15). In expected value, equation (16) becomes the following.

$$E\left[\sqrt{V}\right] \approx \frac{E[V] + V_0}{2\sqrt{V_0}} \tag{17}$$

Equation (17) is also the value of the fair volatility strike price. One may notice that this expected value of the first order approximation is in terms of the value of the fair variance strike price, which is the only volatility derivative whose pricing method is model free according with Brockhaus and Long (2000). Thus, equation (17) shows an interaction between both strike prices.

$$\sqrt{V} \approx \frac{(V+V_0)}{2\sqrt{V_0}} - \frac{(V-V_0)^2}{8(V_0)^{\frac{3}{2}}}$$
(18)

As one may notice from equation (18), the square function in the second term of this approximation might indicate that in expected value, the approximation of equation (15) might rely upon more than just the value of the fair variance strike. As it refers to expectations, one should expect that the second order approximation might result in a more accurate approximation of the square root function. Thus, the expected value of equation (18) is presented in the following statements.

$$\sqrt{V} \approx \frac{(V+V_0)}{2\sqrt{V_0}} - \frac{(V^2 - 2VV_0 + V_0^2)}{8(V_0)^{\frac{3}{2}}}
E[\sqrt{V}] \approx \frac{E[V] + V_0}{2\sqrt{V_0}} - \frac{E[V^2] - 2V_0E[V] + V_0^2}{8(V_0)^{\frac{3}{2}}}$$
(19)

As it can be seen from equation (19), the fair volatility strike value is now on terms of the fair variance strike value and the second order moment of the variance function. This result may be presented in a more elegant statement as it is shown in the next equations.

$$Var(V) = E[V^{2}] - (E[V])^{2}$$
(20)

Equation (20) represents the definition of the variance of any function in terms of the expected value. This definition will be useful in the improvement of the volatility swap value stated in equation (19).

$$E\left[\sqrt{V}\right] \approx \frac{E[V] + V_0}{2\sqrt{V_0}} - \frac{E[V^2] - (E[V])^2 + \{(E[V])^2 - 2V_0E[V] + V_0^2\}}{8(V_0)^{\frac{3}{2}}}$$

$$=\frac{E[V]+V_0}{2\sqrt{V_0}}-\frac{Var(V)+(E[V]-V_0)^2}{8(V_0)^{\frac{3}{2}}}$$
(21)

Equation (21) shows a rather organized approximation of equation (19), in which one can observe now that the second order approximation of the fair volatility strike price is on terms of the fair variance strike price and the variance of the variance. In this case, this approximation is no longer model independent because in order to obtain the variance of the variance one need to make additional assumptions of the underlying volatility process. But, before mention any process with respect to Var(V), the third order approximation will be derived to be used as a pricing method, which represents one of the initiatives introduced by this research.

$$\sqrt{V} \approx \frac{(V+V_0)}{2\sqrt{V_0}} - \frac{(V-V_0)^2}{8(V_0)^{\frac{3}{2}}} + \frac{(V-V_0)^3}{16(V_0)^{\frac{5}{2}}}$$
(22)

Equation (22) shows the third order Taylor expansion. In this approximation one may observe that there is now a cube function in the third term which might represent a challenge for the calculation of the approximation. As it was said for the second order expansion, in this case one should also expect an improvement of the precision of the result.

$$\begin{split} \sqrt{V} &\approx \frac{(V+V_0)}{2\sqrt{V_0}} - \frac{(V-V_0)^2}{8(V_0)^{\frac{3}{2}}} + \frac{V^3 - 3V^2V_0 + 3VV_0^2 - V_0^3}{16(V_0)^{\frac{5}{2}}} \\ E\left[\sqrt{V}\right] &\approx \frac{E[V] + V_0}{2\sqrt{V_0}} - \frac{Var(V) + (E[V] - V_0)^2}{8(V_0)^{\frac{3}{2}}} + \frac{E(V^3 - 3V^2V_0 + 3VV_0^2 - V_0^3)}{16(V_0)^{\frac{5}{2}}} \\ &= \frac{E[V] + V_0}{2\sqrt{V_0}} - \frac{Var(V) + (E[V] - V_0)^2}{8(V_0)^{\frac{3}{2}}} + \frac{E[V^3] - 3V_0E[V^2] + 3V_0^2E[V] - V_0^3}{16(V_0)^{\frac{5}{2}}} \end{split}$$
(23)

Equation (23) shows the expected value of third order expansion of the equation (22). Thus, the fair volatility strike price is now in terms of the value of the fair variance strike price, the value of the variance of the variance and also is in term of the second and third moment of the variance function.

3.3 Exact Result of Volatility Swap

This exact solution is also provided by Brockhaus and Long (2000) in which they show that in order to obtain higher orders of the Taylor expansion there is a function which can be computed by using numerical calculation of multiple derivatives of the Laplace transform of this function. The authors have shown that this function is the following.

$$f(\lambda) = E[e^{-\lambda V}] = Ae^{-\lambda \sigma_0^2 B}$$
(24)

Equation (24) represents de the Laplace transform over the variance function V. In addition, the terms A and B are also functions whose argument is φ , and at the same time, φ is a function itself of some parameters. All these functions are described below.

$$\varphi = \sqrt{\kappa^2 + 2\lambda\gamma^2} \tag{25}$$

$$A = \left\{ \frac{2\varphi e^{(\varphi+\kappa)T/2}}{(\varphi+\kappa)(e^{\varphi T}-1)+2\varphi} \right\}^{2\kappa\theta^2/\gamma^2}$$
(26)

$$B = \frac{2(e^{\varphi T} - 1)}{(\varphi + \kappa)(e^{\varphi T} - 1) + 2\varphi}$$
(27)

Equations (25), (26) and (27) are on terms of the parameters (κ , λ , θ) and γ represents the volatility of volatility. All these parameters come from the stochastic volatility model described by Heston (1993).

$$d\sigma^2 = \kappa(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dZ_t^2$$
(28)

Thus, from equation (28) one may observe that the parameters mentioned before have been obtained from this equation which means that the volatility dynamics for the calculation of higher order Taylor expansion is the one presented by Heston (1993). Alternatively, to compute the value of the fair volatility strike price one may obtain an exact solution, an idea also obtained from Heston (1993), which comes from solving the following integral.

$$E\left[\sqrt{V}\right] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{1 - f(\lambda)}{\lambda^{3/2}} d\lambda$$
 (29)

In equation (29) it is possible to observe that the value of the fair volatility strike price is in terms of an integral⁴ whose argument is an exponential function and is bounded by zero from below and it integrates towards infinity. This results is shown in Brockhaus and Long (2000) and provides with an exact solution that can be used as a benchmark to observe how accurate are the results obtained from the Taylor expansion. From the literature review one may recall that often for traders is more difficult to price volatility swaps than variance swaps. Thus, if they can be provided with a rather simple method, as it is the Taylor expansion, it might be a significant improvement.

Nevertheless, the results obtained by Brockhaus and Long (2000) have shown that, as the volatility of volatility increases the first order approximation shows a poor performance. This performance was expected since the first order approximation is not sensitive to changes in the parameters of the volatility dynamics. The observations indicated that the first order approximation bounded the exact solution from above, and as γ increased the results were even worse. Hence, Brockhaus and Long (2000) have continued with the Taylor expansion by choosing the second order expansion. As it has been said before, this one is no longer model free, which means that a dynamics for the volatility is required to be set. Thus, Heston (1993) dynamics is chosen to compute the second order expansion. In this case, the results of the second order expansion showed the same pattern as the one observed for the exact solution in which the fair volatility strike price tend to decrease at higher values of γ . Despite of the latter statement, it is possible to observe that the second order expansion bounded from below the exact solution. Therefore the approximations bounded from below and above the exact solution as γ increases, hence the authors have proposed that these two approximations may be used as the offer and bid prices for the volatility swap.

3.4 Pricing Volatility Swaps by using Taylor Expansion

In this section it will be explained how the Taylor expansion is utilised. By following Brockhaus and Long (2000), the first approach is to compute the second order of the Taylor expansion. The second approach is to compute the third order of the Taylor expansion whose mechanism has been explained

⁴ This integral is solved numerically by using a adaptive Gauss-Kronrod quadrature provided by Matlab.

in equation (23). Finally, the third method will be also a second order expansion of the equation (15), but there will be a change on the single point to expand around.

As these methods increase in complexity when the order of expansion increases, they will require of more than just the value of the fair variance strike price. Hence, as it has been mentioned early in this section, the equation of the value of the fair variance strike price, the value of the variance of the variance and the second and third moment of the variance function are the following.

$$E[V] = \frac{1 - e^{-\kappa T}}{\kappa T} (\sigma_0^2 - \theta^2) + \theta^2$$
(30)

$$E[V^{2}] = \frac{e^{-2\kappa T}}{2\kappa^{3}} \{2\kappa A + \gamma^{2}B\}$$
(31)

$$E[V^3] = \frac{e^{-3\kappa T}}{2\kappa^5} \{ 2\kappa^2 C + \gamma^4 D - 3\gamma^2 \kappa E \}$$
(32)

$$Var[V] = \frac{\gamma^2 e^{-2\kappa T}}{2\kappa^3 T^2} [2(-1 + e^{2\kappa T} - 2e^{\kappa T}\kappa T)(\sigma_0^2 - \theta^2) + (-1 + 4e^{\kappa T} - 3e^{2\kappa T} + 2e^{2\kappa T}\kappa T)\theta^2]$$
(33)

One may notice that the parameters are the one displayed in equation (28), which are Heston dynamics parameters. Equations (30) and (33) were obtained from the calculations of Brockhaus and Long (2000). However, equations (31) and (32) were obtained from the second and third derivative of the Laplace transform $f(\lambda)$, in which the expressions A, B, C, D and E are on terms of Heston parameters as well, and their definitions may be found in the appendix⁵. One may recall that equation (30) is similar to equation (11) revised from Broadie and Jain (2008).

3.4.1 The Second Order Taylor Expansion

Following the idea of Brockhaus and Long (2000), the single point of the Taylor expansion is equal to E[V]. The argument that the authors have explained is that the value of the fair variance strike price offers a rather reasonable point around which to expand aiming to minimise the upper bound. Thus if one plug the single point chosen into equation (21), the result is the following.

$$E[\sqrt{V}] \approx \frac{E[V] + E[V]}{2\sqrt{E[V]}} - \frac{Var(V) + (E[V] - E[V])^2}{8E[V])^{\frac{3}{2}}}$$
$$= \frac{E[V]}{\sqrt{E[V]}} - \frac{Var(V)}{8(E[V])^{\frac{3}{2}}}$$
(34)

From equation (34), the value of the fair volatility strike price still is in terms of the fair variance strike price and the variance of the variance. In terms of the results⁶, these ones should be the same as the ones obtained in Brockhaus and Long (2000).

⁵ To observe the definition of *A*, *B*, *C*, *D* and *E* refer to the Appendix, Section A.

⁶ All the results obtained from the approximations of the Taylor expansion are calculated in Matlab.

3.4.2 The Third Order Taylor Expansion

This approach represents an extension of the former one, in which the intention is to observe if there is any improvement with respect to the accuracy of the measurement. The expectation is that there should be an improvement regarding the accuracy and therefore the values obtained under this method should be close to the exact solution. Regarding the single point chosen to compute the Taylor expansion, the one for this approach will be the same as above. Thus the single point will be E[V]. Therefore, equation (24) will look like the following expression.

$$E[\sqrt{V}] \approx \frac{E[V] + E[V]}{2\sqrt{E[V]}} - \frac{Var(V) + (E[V] - E[V])^2}{8(E[V])^{\frac{3}{2}}} + \frac{E[V^3] - 3E[V]E[V^2] + 3(E[V])^2E[V] - (E[V])^3}{16(E[V])^{\frac{5}{2}}} = \frac{E[V]}{\sqrt{E[V]}} - \frac{Var(V)}{8(E[V])^{\frac{3}{2}}} + \frac{E[V^3] - 3E[V]E[V^2] + 2(E[V])^3}{16(E[V])^{\frac{5}{2}}}$$
(35)

From equation (35) it is possible to observe that the value of the fair volatility strike price is now on terms of the fair variance strike value, the variance of the variance, the second and the third moment of the variance. Therefore, if equations (30), (31), (32) and (33) are plugged into equation (35), the value of the fair volatility strike will be obtained.

3.4.3 Second Order Taylor Expansion with Different V_0

In this approach, the procedure is similar to the one described on the first method. However, in this case, the single point chosen is different to E[V], the aim is to prove another one that may improve the results obtained by Brockhaus and Long (2000). Thus, the point around which to expand will be the one that makes the third term of the Taylor expansion equal to zero, which is presented in the following equation.

$$g[V_0] = \frac{E[V^3] - 3V_0E[V^2] + 3V_0^2E[V] - V_0^3}{16(V_0)^{\frac{5}{2}}} = 0$$
(36)

In equation (36), the third term of the Taylor expansion is showed as a function of V_0 from which now one needs to find the solution that makes the function $g[V_0]$ equal to zero. The main idea on choosing this particular single point is that one might consider the third term as an adjustment term, and then it might be interesting to find a V_0 that eliminates the effect of this term. The expectation is that there should be an improvement on the calculations with respect to the results obtained from the second order expansion.

3.5 SVI Parameterization and Carr and Madan (1997) Valuation Method

3.5.1 SVI Parameterization

This Stochastic Volatility Inspired or SVI is Jim Gatheral's parameterization presented in his work Gatheral (2006). As it was mentioned early on the literature review, the aim of this parameterization is to observe how the changes on the implied variance surface may affect the final price of the fair

volatility strike calculated by plugging the outcome of this parameterization into Carr and Madan (1997) Valuation method. In other words, options on realised variance will be created and then these ones will be used in the valuation of the fair volatility strike price. In particular, as the parameters of this parameterization may be easily changed, then it will be possible that the shape of the smile adopts the adequate form that fits volatility products. Hence, the parameterization is showed in the following equations.

$$\kappa = \log(K/F) \tag{37}$$

In equation (37), κ is defined as the log of the strike price over the forward price. According to Gatheral (2006) this definition is given by Roger Lee's moment formula. Thus, the range of κ will be the x axis on the graph of the variance surface.

$$Var(\kappa, a, b, \sigma, \rho, m) = a + b\left\{\rho(\kappa - m) + \sqrt{(\kappa - m)^2 + \sigma^2}\right\}$$
(38)

Equation $(39)^7$ represents the volatility parameterization of Jim Gatheral, previously mentioned in equation (6) on the literature review, but it is worth to recall in this section because it will help to the understanding of how this parameterization and Carr and Madan (1997) valuation method are related.

Regarding the explanation of the parameters in equation (38), parameter "a" returns the overall level of the variance; the parameter "b" returns the angle between the left and right asymptotes; the parameter " σ " determines how smooth the vertex is; the parameter " ρ " determines the orientation of the graph; and changes in the parameter "m" translate the graph. An example of the graph is provided below in which the values of the parameters are the same that are displayed in Gatheral (2006).



Figure 1. SVI Parameterization.

⁷ This parameterization is calculated and plotted in Matlab.

In figure 1 one may observe the volatility surface obtained from the parameterization. In this case, the values of the parameters were the ones showed in Gatheral (2006), in which: a = 0.04, b = 0.4, $\sigma = 0.1$, $\rho = -0.4$, m = 0 and $\kappa \in [-1,1]$.

3.5.2 Carr and Madan Valuation Method.

In this subsection the valuation method mentioned in Carr and Madan (1997) will be fully explained as another approach to price volatility products. In particular, they were trying to price these types of products by assuming a static position in options related to those products. The latter statement reflects one key point in the valuation of fair volatility strike price under this approach because options over realised variance are difficult to find on the market, since these ones are traded in OTC markets. For this reason, the SVI parameterization will be helpful in terms of constructing options whose underlying asset will be the realised variance. In terms of the approach itself, Carr and Madan (1997) have assumed that there are no trading opportunities further than the ones that can be made between initial time and time to maturity. In addition, as further assumptions, it is assumed that there exists a future market in risky assets whose delivery time will be greater than the time to maturity. It is also assumed that in the markets there are European style options of all strikes. The latter assumption is similar to assume that there is a continuous trading in the market, but one may notice that assume a continuum of strike price is far from reality. Although the latter statement might be true, there options like future options on the S&P500 which are highly close to match that particular assumption. Regarding the approach, the authors have said that market structure assumed allow any investor to build any smooth function of the terminal price of the future by assuming a static position in options at initial time. This function is the following.

$$f(F_T) = f(k) + f'(k)[(F_T - k)^+ - (k - F_T)^+] + \int_0^K f''(K)(K - F_T)^+ dK + \int_K^\infty f''(K)(F_T - K)^+ dK$$
(39)

Equation (39) represents the valuation stated by Carr and Madan (1997). In this equation, the first term may be interpreted as the static position payoff of f(k) pure discount bonds, in which each one of them should pay one dollar at time to maturity according with the description of the authors. In addition, the second term may be assumed to be the payoff of calls with strike price k less than puts with same strike price times f'(k). Finally, the third and the fourth terms represent a static position of put and call options respectively in which both of them are struck at K and are multiplied by the function f''(K).

Now, in terms of the analysis, the argument of the function described on equation (39) is a future price. However, in this case the argument will be the realised variance. Thus, the function itself will be defined as the following.

$$f(V) = V^{\frac{1}{2}}$$
(40)

Equation (40) represents the function that will be used in the analysis conducted on this sub section. In addition, as stated below, the value of k will be equal to the value of the fair variance strike price, which creates a relationship between the fair volatility and variance strike values as showed in Brockhaus and Long (2000).

$$k = E[V] \tag{41}$$

Now, by plugging equations (40) and (41) into equation (39), this one will turn into the following equation.

$$V^{\frac{1}{2}} = (E[V])^{\frac{1}{2}} + \frac{1}{2(E[V])^{\frac{1}{2}}} [(V - E[V])^{+} - (E[V] - V)^{+}] + \int_{0}^{K} -\frac{1}{4(E[V])^{\frac{3}{2}}} (K - V)^{+} dK + \int_{K}^{\infty} -\frac{1}{4(E[V])^{\frac{3}{2}}} (V - K)^{+} dK$$
(42)

Thus, equation (42) defines the methodology taken from Carr and Madan (1997) which will be used to price fair volatility strike. In order to obtain the value of this one, it is necessary to apply expected value to equation (42). Hence, if the expected value is applied, equation (42) will turn into the following one.

$$E\left[V^{\frac{1}{2}}\right] = \left(E[V]\right)^{\frac{1}{2}} - \frac{1}{4(E[V])^{\frac{3}{2}}} \left\{ \int_{0}^{K} (K-V)^{+} dK + \int_{K}^{\infty} (V-K)^{+} dK \right\}$$
(43)

By the result of equation $(43)^8$, the value of the fair volatility strike price may be computed, and one will be able to observe the changes of this one due to changes in the volatility skew or in the smile curve of the implied variance function.

Equation (43) requires of options over realised variance, obtained from the SVI parameterization combined with Black (1976) formulae for option pricing. In addition, the strike price will be obtained from equation (37) by applying the exponential function to this one. Hence, these inputs must be plugged into the following equations.

$$c = e^{-rT} \{ FN(d_1) - KN(d_2) \}$$
 (44)

$$p = e^{-rT} \{ KN(d_2) - FN(d_1) \}$$
(45)

Equation (44) and (45) represents the call and put valuation formulae obtained from Black (1976) respectively. It is important to notice that the forward price in this case is the fair variance strike price, which is why these options are on realised variance. Once that the values of these options are obtained⁹, one just need to plug them into equation (43) and then the fair volatility strike price will be obtained.

4 Results

4.1 Taylor Expansion Results

In Brockhaus and Long (2000) the results of the Taylor expansion are presented in such a manner that the authors compared the exact solution with the first and second order expansion against different values of the volatility of volatility or gamma. In this case, the same pattern will be selected in order to be able to compare this results with the ones obtained from the mentioned authors. Hence the results obtained from the Taylor expansion and the exact solution will be showed in the following graph.

⁸ The results of this equation are obtained in Matlab, by solving the integral with the Trapezium rule.

⁹ The values of these options are calculated in Matlab.



Figure 2. Taylor Expansion Approximations and Heston Exact Solution.

In figure 2^{10} one might observe that here is a plausible difference among each one of the approaches selected. First of all, the values of the parameters chosen for these calculations are the followings:

Parameters	
κ	3.0
θ	0.5
σ_0	0.5
Т	1.0
γ	[0.01,1.5]

Table 1. Values of Parameters of Heston Dynamics.

The values of table 1¹¹ were obtained from a calibration performed on Bloomberg of the Heston model for VIX options. In addition, one may found similar results regarding the values of the parameters in Sepp (2008), in which the author also calibrates Heston model to price Options on realised variance.

Before entering in a deeper one by one analysis, it is worth to explain why all of them are decreasing while gamma is increasing. In particular, there is one explanation of why the negative slope observed for all the methods in figure 2, and is because the majority of the expressions are in terms of the inverse of a square root function. One might say the higher the volatility of volatility the more disperse is the distribution, which in consequence increases the effect of the volatility convexity.

Now, in terms of the results of each method, the shape of the curves of the exact solution and the second order expansion are almost as expected. With respect to the latter, both approaches evidenced a similar performance as Brockhaus and Long (2000) showed in their figure. One might say that the

¹⁰ One may observe that in the y axis of figure 2 it says "Volatility Swap". Although this label should be "Fair Volatility Strike Price", is presented as above in order to keep the line presented by Brockhaus and Long (2000).

¹¹ Gamma runs from 0.01 to 1.5 by increments of 0.01.

main difference remain in that the shape of both curves decay faster, but it is important to notice that there is not much difference among the values of the fair volatility strike yielded by these two methods, and the second order expansion is still underneath the exact solution as displayed by the authors. Regarding the second order expansion with different single point, its result is as much as fair. It was expected to obtain an improvement in the precision but it seems that this one is going farther from the exact solution, overestimating it as gamma increases. This unexpected result might be explained in terms of the expansion method, in which Taylor approximation might not be converging towards the exact solution. Furthermore, it might be possible that this different V_0 may cause a positive impact from one of the terms of the expansion.

With respect to the third order approximation, its outcome seems rather suspicious and highly unexpected. In particular, it was expected that this one performs reasonable above from the result of the second order expansion, but it turned out to be the other way around. One possible explanation of this performance might be found in terms of the parameters, in this case the volatility of volatility. Brockhaus and Long (2000) have mentioned that gamma has the effect of diluting the reversion speed, therefore it seems that the value of the volatility swap given from the Taylor expansion approximation will not converge to its exact solution as one increase the order of the approximation. It could be said that the effect of gamma is strengthened when the order of the expansion is increased, and also when kappa is small and theta and sigma are equal, which are the long term and short term variance respectively. Another strong possibility is that the coefficient in front of each term of the expansion increases as the order of the expansion grows very large. Therefore when they indeed become large, then the value of the fair volatility strike price will tend to diverge from the value of the exact solution. Finally, another possible explanation of why the third order expansion performs worse than the second order expansion is that it is possible that the realised volatility might have to be within a certain range. Thus as one increases the value of gamma the realised volatility might lay too much away from this range, which implies that the Taylor expansion should diverge.

The latter paragraphs mentioned some possible explanations of the outcome of figure 2. In particular the explanation related of the effect of gamma might be explained further as some figures might be helpful to understand this effect. Thus, in order to try to offset the diluting effect of gamma one might increase the value of kappa since this parameter represents the speed of reversion. The results are showed in the following figure.



Figure 3. Taylor Expansion Approximation and Heston Exact Solution with Different ĸ.

In figure 3 one might observe that the results have improved with respect to the ones showed in figure 2. As it was mentioned in the last paragraph in order to offset the diluting effect of increasing gamma it is necessary to increase the value of kappa. Well in figure 3 the value of kappa has been changed to be nine, the triple of the first one selected, the rest of the parameters remained with the same value as showed in table 1. If one notice the value of the fair volatility strike price in the y axis, these ones are highly closer each one another. In terms of the slope of each curve, again the same pattern of figure 2 is obtained here, which is an observation that is not likely to change due to the square root function. Regarding each pattern separately, they have not changed in terms of their tendency. In addition, the second order expansion with different single point is yet again over the value of the exact solution, and the reasons are likely to be the same as mentioned before.

Thus, from figure 3 the conclusion that one might state is that the higher the value of kappa the more accurate are the solutions obtained from each methodology, though the tendency showed in figure 2 remains the same. This result tends to reinforce what has been said above in terms that gamma was causing the results to be that dispersed. Since this parameter reduces the speed of reversion of this process, then the manner to offset this tendency is by increasing the reversion speed reflected in the parameter kappa.

So far the finding that has resulted farther from expectations is the third order of the Taylor approximation. Hence, it might be worth to focus on the result of this approach separately and contrast it with the exact solution. In order to do so and to improve the quality of the analysis these two approaches will be plotted against different values of each parameter with the exception of gamma since this one has been already described. Thus, in the next figures it will be possible to observe the performance of the third order approximation against changes on the parameters.



Figure 4. Volatility Swap Value Against ĸ.

Figure 4 represents one of the first attempts to show the performance of the third order expansion. In this case, the values of the parameters are:

Parameters	
κ	[0.1,15.0]
θ	0.5
σ_0	0.5
Т	1.0
Y	1.0

Table 2. Values of the Parameters When κ Changes.

As one might observe from figure 4 and table 2, kappa¹² is the parameter in the x axis instead of gamma, and the latter has been chosen to be equal to one since at this level each of the methodologies has shown different values of the fair volatility strike price. In terms of the result of figure 4 it can be said that the third order expansion improves its performance as kappa increases. The latter statement is in line with what has been said before regarding the effect of kappa in the value of the volatility swap from each method. As kappa represents the reversion speed, if this one increases then the value of the variance will reach its long term value faster. Therefore the results from the third order expansion will match the ones from the exact solution whenever the reversion speed is higher. In this case, at a level of volatility of volatility equal to one, the third order expansion approximately matches the exact solution at value of kappa equal to eight. Furthermore, this result may be also observed in figure 3 in which the value of kappa is equal to nine, and at a value of gamma equal to one, there is relatively no difference between the exact solution and the third order expansion.

¹² Kappa runs from 0.1 to 15.0 by increments of 0.1.

In order to continue with the analysis, it might be interesting to observe how well performs the third order expansion when the parameter that defines the long term level of variance changes.



Figure 5. Volatility Swap Value Against θ .

In figure 5^{13} the results of the third order expansion and the exact solution are almost the same. This particular outcome might be explained in terms of the level of the variance. As the value of theta increases, the value of the fair volatility strike price must increase as well, since this one is affected directly by the level of the variance. Therefore, it can be said that the higher the long term value of the variance the higher will be the value of the fair volatility strike price. In addition, it might also be said that higher values of theta could offset the diluting effect of gamma. In order to prove the theory of the latter statement, the next figure will show the results of all of the approaches against gamma with a higher value of theta.

¹³ In figure 5, the value of the parameters are the same as showed in table 2 with the exception of the value of kappa which in this case is equal to 3.0, and the value of θ that runs from 0.1 to 15.0 by increments of 0.1.



Figure 6. Taylor Expansion Approximations and Heston Exact Solution with Different θ .

From figure 6 one might observe the effect of increasing theta to five while sigma and the rest of the parameters remained the same value mentioned in table 1. As has been said before, it seems that the diluting effect of gamma disappears even for high values of this parameter. Regarding the value of the fair volatility strike price, the range is wider than the one resulting from increasing kappa in figure 3. The latter might be due to the fact that the short term variance has not been changed, then the difference between the long and short term variance increases with higher values of theta, resulting in a minor value of the fair volatility strike price. In the appendix section¹⁴ it will be shown a figure in which both parameter are changed equally.

Now, regarding the analysis of the third order approximation, the following figure will show its performance when the maturity is changed.

¹⁴ Appendix section: Figure 15.



Figure 7. Volatility Swap Value Against Maturity.

In figure 7¹⁵, the findings are rather interesting. The results of both approaches are not that far, and even more they become closer as maturity increases. What is interesting of figure 7 is the shape of both curves in which at beginning both tend to decrease until they reach maturity equal to one, and then the pattern is reversed. This interesting patter might be caused by the term in front of each one of the approaches, in which the main one is the maturity multiplied by an exponential function whose arguments are kappa multiplied by maturity. In order to observe how the maturity impact the other approaches, a graph in which all the approaches are plotted against gamma with a maturity value higher than the one showed in table 1 is displayed on the appendix¹⁶.

In order to observe if there is an improvement in the outcome of figure 7, one might increase the value of kappa. The aim of the latter exercise is to enhance the measurement of the third order approximation and also to examine if there is any change on the shape of both functions. Thus, the following figure shows the same approaches as in figure 7 but with a higher value of kappa.

¹⁵ The value of the parameters are the same as in table 2, with the exception of gamma which is equal to 1.0 and the time to maturity, which in this case runs from 0.1 to 15.0 by increments of 0.1.

¹⁶ Appendix, Section B: Figure 16.



Figure 8. Volatility Swap Value Against Maturity with Different Value of *k*.

Figure 8 shows the value of the fair volatility strike obtained from the third order expansion and the exact solution. In this case the parameters are the same as in figure 7 with the exception of kappa in which now the value has been increased to fifteen. The latter seems to be an extremely increase in the value of kappa, but as one might recall from figure 4, at that particular level there is a good convergence of the third order expansion towards the exact solution. Regarding the outcome of figure 8 it is possible to observe that here is a notorious change in the shape of both curves. Now there is not a point of inflection, both functions seem to increase as the maturity increases with marginal decreases at each level of maturity. The patterns showed in figure 8 are what were expected before, thus as the reversion speed was increased the impact of the front terms of the approaches has been mitigated. The latter statement represents a possible conclusion in which again the theory that higher order of the Taylor expansion performs better with higher values of reversion speed, and in this case, with higher values of maturity as well.

All the results presented so far with respect to the third order expansion are also applicable to the second order expansion and to the second order expansion with different V_0 . These one should also perform better with higher values of the reversion speed and maturity as well, and with lower values of the volatility of volatility. Graphs in which both function are displayed along with the exact solution against changes on the different parameters are showed in the appendix section¹⁷.

4.2 SVI Parameterization and Carr and Madan Valuation Method Results

From the methodology section one may recall that these two approaches were combined aiming to price the fair volatility strike under a completely different approach. In particular, it might be rather interesting to observe how changes in the parameters of the SVI parameterization could affect the resulting price of the fair volatility strike price. Furthermore, it will be possible to see how the changes in the volatility skew and implied variance smile affect this fair strike price. Thus, in the next figure

¹⁷ Appendix, Section B: Figure 17, Figure 18, Figure 19 and Figure 20.

one may observe the shape of the curve obtained from the SVI parameterization and also the value of the implied variance and resultant price of the fair volatility strike.



Figure 9. Stochastic Volatility Inspired Parameterization.

In figure 9, the parameterization showed is obtained by using the following values of the parameters:

Parameters	
a	0.4
b	0.4
ρ	0.4
m	-0.3
σ	1.1
κ	[-3.0,3.0]

Table 3. Parameters Values of the SVI Parameterization.

The values of table 3¹⁸ were selected as such because these ones may generate an implied variance smile that fits the one obtained from the market. Although the latter statement refers to fit the parameters to the implied variance of the market, there are not data available from the market regarding options on realised variance. However, the data from options over the volatility index of the S&P 500, the VIX index¹⁹, generates this type of shape with respect of the implied variance. Therefore, the VIX Index has been used as a model to derive an appropriate shape for the implied variance from options on realised variance. In addition, regarding these types of options, the implied variance as a function of the strike price, tend to increase when the strike price increases as one might observe in equation (38) in the methodology section.

¹⁸ Kappa runs from -3 to 3 by increments of 0.01.

¹⁹ Appendix, Section C: Figure 21, the VIX Index Implied Variance obtained from Bloomberg.

Once that the SVI parameterization results have been described, it is worth to mention the final results from the Carr and Madan (1997) methodology. As it was explained in section 2 and 3, the connexion of Gatheral's approach and this one is that the former provide the volatility and the strike price to the options on realised variance that will be used in the static portfolio required. In addition, it might be good to recall that the function to be used in Carr and Madan (1997) approach is the square root of the variance. Thus, given that the expected value of the variance, or the fair variance strike value, will be the one obtained from the first moment of the variance with Heston parameters, which is: E[V] =0.0250; the result of the valuation is: $E\left[V^{\frac{1}{2}}\right] = 0.1405$. Now, in order to enhance the analysis of the results it might be an excellent exercise to change the value of the parameters from the SVI parameterization and observe how these changes affects the value of the fair volatility strike price. With respect to these parameters and their meaning, all of them have been already described in the methodology section and each of them contribute to change the shape of the implied variance smile. One important feature of this approach to be highlighted is that in this case the volatility skew and the smile of the distribution are not assumed to be fixed as it is in Heston. The latter represents a rather important difference about these two valuations forms, in which the ones based on Heston parameters have assumed a given implied variance surface whereas in the present one, the skew is free to be changed along with the smile by changing the parameters. Thus, the following figure will show how a change of one of the parameters might affect the volatility skew, the smile, and also the value of the fair volatility strike price.



Figure 10. Implied Variance from SVI with Different Value of b.

Before discussing the results from figure 10, it is worth to recall that the parameters that might affect the smile and the volatility skew are "b" and " ρ " respectively. Thus, in the case of figure 10 the former parameter has been changed in which its value now is: b = 0.1. The rest of the parameters remained the same as described in table 3. By recalling the definition of "b", this one is responsible of the angle between the left and the right asymptotes, which means that the level of the smile should not be affected. However, from figure 10, one might observe that the latter statement was not totally correct

since the level of the smile has been affected by the change on the parameter "b", though the wings of the smile have also been extended as it was expected. The value of the implied variance obtained in IV_1 , when kappa is at the money, is $\sigma_{im}^2 = 0.526$. Whereas the value of the implied variance obtained in IV_0 , which is the case of figure 9 is $\sigma_{im}^2 = 0.9041$, when kappa is also at the money. Therefore there is an inconsistency with regards to the role of the parameter "b", which it might distort any conclusion that one might want to provide with respect to the effects that the fair volatility strike price may experience when the smile and the skew are changed. In order to fix this problem, more than one parameter will be modified at the same time, which in this case are likely to be the parameter "b", and the parameter "a" whose function is to control the overall level of the variance, as defined in the methodology section. Hence, whenever "b" is changed so it must be for the parameter "a", in order to keep the same level of the implied variance smile. If the level of the implied variance is kept, and if the wings of the smile become wider or closer, then it will be possible to state any observation regarding the interaction of the smile of the implied variance and the resulting value of the fair volatility strike price. Thus, the following figure will illustrate the interaction of the parameters, the resultant smile, and the final value of the fair strike price.



Figure 11. Implied Variance with Different Combination of a and b.

In figure 11 it is possible to observe the variations on the wings of the smile of the implied variance function. Before entering in a further discussion of the implications of figure 11, the next table summarize the values employed in each one of the smiles showed above.

(a,b)	
(a0,b0)	(0.29,0.5)
(a1,b1)	(0.4,0.4)
(a2,b2)	(0.51,0.3)
(a3,b3)	(0.62,0.2)
(a4,b4)	(0.73,0.1)

Table 4. Pair of Values for the Parameters a and b.

The values displayed in table 4 are the combinations of the parameter "a" and "b" employed to generate each one the curves showed in figure 11. As it was stated before, it is necessary to set a combination of both parameters to keep the same level when the volatility is at the money. In this case, to keep the value of the implied variance resulted from the at the money volatility, the values of the parameters mentioned in table 3 where used with the exception that "m" and " ρ " were set to be equal to zero. The latter values were selected as such in behalf of the analysis. Regarding the value of the implied variance at kappa equal to zero, this one is 0.84, and as one might observe from figure 11, this level is fully achieved by each one of the smiles. This level of implied variance was set to be as such because the first values of the parameters "a" and "b" yielded that level of implied variance, thus naturally that value was kept as the base level.

Now, in terms of the value of the fair volatility strikes at each level of the smile, these ones are summarised in the following table.

IV(a, b)	Fair Volatility Strike
$IV(a_0, b_0)$	0.1410
$IV(a_1, b_1)$	0.1416
$IV(a_2, b_2)$	0.1422
$IV(a_3, b_3)$	0.1428
$IV(a_4, b_4)$	0.1434

Table 5. Fair Volatility Strike Price and Different Smile functions.

In table 5 it is possible to observe the values of the fait volatility strike price at different level of the smile. When the wings of smile are more vertical as it is shown by $IV(a_0, b_0)$ then the resultant value of the fair volatility strike is lower than when the wings of the smile are rather horizontal. Thus, from this experiment one might conclude that changes on the wings of the smile do affect the value of the fair volatility strike price, and hence it might be important to focus on this feature if one wants to price volatility swaps without any error. The importance of this finding is that it might challenge the procedures based on Heston volatility dynamics stated at the first sub section of the methodology, in which all of them works under the assumption of a fixed smile function. Thus, in order to confirm the findings of figure 11 and table 5, it is necessary to conduct more research. In order to do so, there is one parameter that affects the volatility skew, as mentioned before, " ρ " determines the orientation of the smile or the skew, without changing the level of the variance. Well, the latter statement will be put to test, as it happened before, it might not be as it says. In other words, there is a probability that " ρ " not only changes the orientation of the smile but also the level of the variance.

The next figure will present the results of changing the parameter " ρ ". Again it will be presented five different measurements so one might observe properly any difference.



Figure 12. Implied Variance at Different Values of ρ .

Figure 12 shows the different implied variance functions obtained from different values of the parameter " ρ ". One may observe that the level of the variance, when the strike price is at the money, remains the same, at the same set of values of the parameters "a" and "b" displayed in table 4. Thus, in this case, changes in the parameter " ρ " did only changed the orientation of the graph. The values of the latter parameters are presented in the following table.

ŀ	0
ρ	0.0
ρ_1	0.1
ρ_2	0.2
ρ_3	0.3
ρ_4	0.4

Table 6. Different Values of Parameter ρ .

And the results with respect to the volatility swap value are shown in the following table.

IV	(p)	Fair Volatility Strike
IV(ρ₀)	0.1416
IV((ρ_1)	0.1416
IV(ρ_2)	0.1416
IV(ρ ₃)	0.1416
IV((ρ_4)	0.1415

Table 7. Implied Variance with Different ρ and Fair Volatility Strike Price.

The values of the parameter " ρ " showed in table 6 produced the values of the fair volatility strike price displayed in table 7. Then, one might say that changes on the volatility skew my not produce significant changes on the fair volatility strike price. However, one should not state that this one is

insensitive to changes on the skew. On the contrary, it seems that it might need of a rather significant change in the value of that parameter to obtain a reasonable change in the value of the fair volatility strike price. Thus, in the next figure it will be shown the effects of make a large change in the value of parameter " ρ ".



Figure 13. Implied Variance at two Different Values of ρ .

From figure 13 one may notice a large change in volatility skew produced by a considerable change in the parameter " ρ ". In this case, the mentioned parameter has been changed from being equal to zero, to be equal to one, in which ρ^* represents the latter option. This change, which should be considered as a large one if one observe the variation on the implied variance curve, has not produce a considerable alteration on the value of the fair volatility strike price. The latter has changed from being 0.1416 to be equal to 0.1409, which is not a large modification considering the variation of the curves in figure 13. However, one can state that the fair volatility strike price is at some level sensitive to changes on the volatility skew. Nevertheless, one might said that the fair volatility strike price is more sensitive to changes on the wings of the smile of the implied variance function showed in figure 11 than to changes on the volatility skew displayed in figure 12.

There is one parameter that might be worth to observe its influence in the value of the fair volatility strike price. This parameter is " σ " whose definition, provided in the last section, is that this one determines how smooth the vertex is. Thus, it might be a rather interesting exercise to observe the implications of changes in this parameter over the resultant value of the fair volatility strike price and to compare it with the results obtained from figure 11. In order to do so, and to keep the level of the implied variance the same when kappa is at the money, the parameter "b" will be also modified. Thus it will be possible to observe just changes on the wings of the smile of the implied variance function as shown in figure 11. Hence, the implied variance smile function will be showed in the following figure.



Figure 14. Implied Variance Values at Different Values of σ .

The outcome of figure 14 is reasonable similar to the one obtained in figure 11. As one might observe, the level of the implied is the same for each of the attempts, but the smile function is different in each one of them. Before mentioning the yielded values of the fair volatility strike price, it is worth to state the combination of the parameters utilised in this case. Thus, the following table will display the values of " σ " and "b".

(<i>o</i>	(o , b)	
(σ_0, b_0)	(0.9,0.4889)	
(σ_1, b_1)	(1.2,0.3667)	
(σ_2, b_2)	(1.6,0.2750)	
(σ_3, b_3)	(2.0,0.2200)	
(σ_4, b_4)	(2.6,0.1692)	

Table 8. Combination of Values of Parameters σ and b.

In addition, the next table will show the final values of the fair volatility strike price given the new set of parameters defined in table 8.

$W(\sigma, b)$	Fair Volatility Strike
$IV(\sigma_0, b_0)$	0.1406
$IV(\sigma_1, b_1)$	0.1420
$IV(\sigma_2, b_2)$	0.1428
$IV(\sigma_3, b_3)$	0.1431
$IV(\sigma_4, b_4)$	0.1434

Table 9. Implied Variance with Different σ and Fair Volatility Strike Price.

Thus, the set of parameters mentioned in table 8, which were chosen to generate a similar graph as figure 11, produced the values of the fair volatility strike price provided in table 9, assuming the same value for the rest of the parameters in which "a" is equal to 0.4 and " ρ " is equal to 0. It might be

observed that the value of the fair volatility strike price increases at almost the same rate as in table 5. Thus, one might state that this new set of parameters produce the same effect as the set of parameters chosen on figure 11, in which changes on the wings of smile indeed produce a change on the final fair volatility strike price.

As it can be seen in figure 11 and in figure 14, the wider the wing of the variance smile are, the higher the value of the fair volatility strike price will be. One could say that the more the options is far from the at the money point the higher will be the value of the fair volatility strike price according with the approaches selected to perform the measurement. As final remarks, through all the changes in the parameterization, it has been possible to observe that changes in the wings of smile of the implied variance function indeed affect the final result of the fair volatility strike price, while changes in the skew did not seem to affect the fair volatility strike. Hence, one might have to be careful in pricing volatility swaps by assuming a constant surface of the smile.

5 Limitations and Future Research

Regarding the limitations one might observe that during this work some problems have arisen through this one. One of them is related to the Taylor expansion and its convergence property. One should expect that the higher the order of the Taylor expansion the better should be the convergence achieved, in which by consequence, the accuracy of the result should be improved. Recalling from the last section, the results showed that a higher order of the Taylor expansion yielded a worse outcome than the lower order of such expansion. Therefore, in this particular case, this approach diverges as the order of the expansion was increased²⁰. Another limitation is related to the Carr and Madan (1997) valuation method utilised for pricing the fair volatility strike price. This approach requires of static portfolio in options as inputs of the calculation of the final function. Therefore, if one is working with the expected value of the square root of the variance as the function, then the options must be over realised variance. The latter statement represents the limitation in this case, in which options whose underlying asset is realised variance are not available in some financial platforms like Bloomberg or the CBOE website²¹.

Regarding future research, there is one approach in particular that seems rather motivating to continue. This one is related to the volatility dynamics used to price volatility swaps and variance swaps. Based on the work of Brockhaus and Long (2000), Heston volatility dynamics has been applied to calculate the values of the far variance and volatility strike price. Since all the calculations were done based on Heston's parameters, it might be possible that this process may cause the divergence presented in the Taylor expansion approach described before. Although the latter might be seen as a limitation, is rather an interesting opportunity to expand this research. Thus, further researchers should try another volatility dynamics or similar process like Heath – Jarrow – Morton described in Heath et al. (1992). This framework may model the dynamics of instantaneous forward rates, which is an idea applied by other authors like Dupire (1993), who modelled the random evolution of the term structure of forward variances over time.

²⁰ This cannot be generalised for higher orders since the approach used only tried with the third order. The reader may notice that obtaining terms of higher orders of this expansion might represent a hard task.

²¹ "Chicago Board Options Exchange" www.cboe.com

6 Conclusions

Through this work, volatility swaps and variance swaps have been explained in detail, in which full descriptions of their characteristics and their pricing approaches have been provided. In this line, some of the expectations of these pricing methods have been achieved, while some of them have not been reached. Furthermore, it was expected that higher order of the Taylor expansion will guarantee an enhancement on the convergence towards the exact solution. The results showed that in this case the third order of the Taylor expansion did not improve the approximation, evidencing that in some cases this expansion does not approximate better as one increase the order. Thus, in this exercise, the second order expansion provided an accurate result than the third one. This particular result came as a surprise and it represented a challenge in terms of possible explanations that one could state in order to understand it. As some ideas were listed, this unexpected scenario might be explained in terms of the value of the parameters and in the form of the expressions of the Taylor expansion. At the end, one might conclude that for some values of the Heston parameters higher order of the Taylor expansion will not be more useful than the lower order of this approximation. One of these parameters is the volatility of volatility whose dilutive property might have influenced the poor performance of the third order Taylor expansion. Another explanation provided was that the term in from of each of the terms of the approximation grew higher when the order was increased which might result in that the higher order terms will affect more the final result of the approximation.

Regarding the second methodology, the aim was to show how the smile surface and the volatility skew observed in an implied volatility function parameterization might affect the final value of the fair volatility strike price. As mentioned, it was possible to observe such relationship because the SVI parameterization and the Carr and Madan (1997) valuation method were combined. The result of this combination showed that the wings of the smile may affect the value of the volatility swap, whereas the skew has a lower impact on this derivative. Thus, one can contrast this finding with the assumption that the smile surface and the volatility skew are constant in Heston process. Therefore the values obtained from that volatility dynamics process might be biased. In other words, if one would attempt to price volatility swaps under Heston dynamics then an adjustment or correction by changes either the smile surface or the volatility skew should be carry out in order to obtain better results. Although the latter recommendation might be useful, the impacts of changing both parameters are rather modest given the results obtained.

At this point, it is worth to highlight that every result obtained in this work has been calculated over a continuous finance time which is considerable unrealistic. It is likely that the results might be biased when a discretisation is placed, in this case, these ones might be even worse, especially the ones obtained from the Taylor expansion. Thus, one should be careful in directly applying any conclusion of this present work to the real world.

Therefore, more work still can be done in order to keep improving these pricing approaches. It will be interesting to observe a thorough research regarding the effects of the smile of an implied variance function on volatility derivatives. In summary, future researchers are encouraged to continue with investigations on volatility derivatives, since these assets are playing a major role in these times of financial turmoil.

7 References

Black, F. (1976) The Pricing of Commodity Contracts. Journal of Financial Economics, 3, 167 – 179.

Broadie, M. & Jain, A. (2008) Pricing and Hedging Volatility Derivatives. Fall 2006 Informs Annual Conference, London Business School.

Brockhaus, O. Long, D. (2000) Volatility Swaps made Simple. Technical Article, Risk Magazine.

Carr, P. & Lee, R. (2007) Realised Volatility and Variance Options via Swaps. Cutting Edge Volatility Options, Risk Magazine.

Carr, P. & Lee, R. (2009) Volatility Derivatives The Annual Review of Finance. Econ. Annual Reviews.

Carr, P. & Madan, D. (1997) Towards a Theory of Volatility Trading. Working paper, December 16, 1997.

Clark, J. (2010) Return to Variance?. Equity Derivatives Article, Risk Magazine.

Cox, J. Ingersoll, J. Ross, S. A Theory of the Term Structure of Interest Rates. Econometrica, Volume 53, 385 – 407.

Demeterfi, K. et al. (1999) Quantitative Strategy Research Notes: More than You Ever Wanted to Know about Volatility Swaps. Goldman Sachs & Co.

Dupire, B. (1993) Model Art Risk. Risk September, 118 – 120.

Gatheral, J. (2006) The Volatility Surface A Practitioner's Guide. John Wiley & Sons, Inc.

Heath, D. Jarrow, R. And Morton, A. (1992) Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. Econometrica, 60, 77 – 105.

Heston, S. (1993) A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. The Review of Financial Studies, Volume 6, Issue 2 (1993), 327 – 343.

Mougeot, N. (2005) Volatility Investing Handbook. Equity Derivatives Technical Study, Equity & Derivatives Research, BNP Paribas.

Neuberger, A. (1994) The Log Contract a New Instrument to Hedge Volatility. Journal of Portfolio Management, Winter (1994), 74.

Pengelly, C. (2009) An Aversion to Variance. Variance Swaps Article, Risk Magazine.

Sepp, A. (2008) Pricing Options on Realised Variance in the Heston Model with Jumps Returns and Volatility. The Journal of Computational Finance, Volume 11, Number 4 (2008), 33 – 70.

8 Appendix

Section A

From equations (31) and (32), the definitions of A, B, C, D and E are the followings:

$$\begin{split} A &= \left[(-1 + e^{\kappa T}) \sigma_0^2 + \left(1 + e^{\kappa T} (-1 + \kappa T) \right) \theta^2 \right]^2 \\ B &= \left[2 \sigma_0^2 (-1 + e^{2\kappa T} - 2 e^{\kappa T} \kappa T) + \left(1 + 4 e^{\kappa T} (1 + \kappa T) + e^{2\kappa T} (-5 + 2\kappa T) \right) \theta^2 \right] \\ C &= \left[(-1 + e^{\kappa T}) \sigma_0^2 + \left(1 + e^{\kappa T} (-1 + \kappa T) \right) \theta^2 \right]^3 \\ D &= \left[3 \sigma_0^2 \left(-1 + 2 e^{3\kappa T} - 2 e^{\kappa T} (1 + 2\kappa T) + e^{2\kappa T} (1 - 2\kappa T - 2\kappa^2 T^2) \right) \\ &+ \left(1 + 6 e^{\kappa T} (1 + \kappa T) + e^{3\kappa T} (-22 + 6\kappa T) + 3 e^{2\kappa T} (5 + 6\kappa T + 2\kappa^2 T^2) \right) \theta^2 \right] \\ E &= \left[-2 (-1 + e^{\kappa T}) \sigma_0^4 (-1 + e^{2\kappa T} - 2 e^{\kappa T} \kappa T) \\ &- \sigma_0^2 \left(-3 + e^{3\kappa T} (-7 + 4\kappa T) - e^{\kappa T} (1 + 10\kappa T) + e^{2\kappa T} (11 + 6\kappa T - 4\kappa^2 T^2) \right) \theta^2 \\ &- \left(1 + e^{\kappa T} (3 + 5\kappa T) + e^{3\kappa T} (5 - 7\kappa T + 2\kappa^2 T^2) + e^{2\kappa T} (-9 + 2\kappa T + 4\kappa^2 T^2) \right) \theta^4 \right] \end{split}$$

Section **B**



Figure 15. Taylor Approximations and Heston Exact Solution with Higher Values of θ and σ_0 .

In figure 15 one may observe that the pattern described by all the approaches is the same as the obtained from figure 6, but the difference remains in the range of values of the fair volatility strike price. While in the latter figure the range is broader, in the former figure the range is closer. Even more, the convergence is of two decimal points. This improving on the accuracy of the results might be due to the fact that in figure 15, the long term variance (θ) and the short term variance (σ_0) are of equal value and this value is higher. In this case the value chosen for these two parameters is equal to five. Therefore, one might say that as both variance parameters are increased at equal value, the accuracy of all of the approaches is higher.



Figure 16. Taylor Approximations and Heston Exact Solution with Higher values of Maturity.

In figure 16, all the approaches are plotted against gamma in which the maturity has been increased from being one year to be ten. This exercise attempts to show if there is any improvement in the results when one increases the maturity while the rest of the parameters remained the same as in table 1. If one compares the results obtained here with the ones obtained in figure 2, there is an improvement in the values of the fair volatility strike price. The range of values of this one is small in comparison with the range obtained in figure 2. Therefore, it may be stated that as one increases the maturities the convergence becomes more accurate. Further, this result may imply that the higher the maturity is, the less the diluting effect affects the results as it was stated in the conclusions.



Figure 17. Different V_0 Approximation and Heston Exact Solution at Different Values of κ .

From figure 17 one may observe that the outcome is as expected. The different V_0 bound the exact solution from above, but as the value of kappa increases, this difference becomes smaller. This pattern is the same for the third order approximation, and even the pattern obtained for the different V_0 is the same as in figure 2, in which this one is always above the exact solution. In this case, the same conclusion may be stated, in which as one increases the values of kappa the results are improved for the same reasons stated in the results section.



Figure 18. Different V₀ Approximation and Heston Exact Solution at Different Values of Maturity.

The shape of the curves obtained in figure 18 is the same as the one displayed in figure 7, in which the main difference remain in that the different V_0 approach bound the exact solution from above. This pattern is as expected, as in figure 17 and others, the different V_0 approach always bounded the exact solution from above. In addition, as one increase the value of the maturity the results of each of the approaches become closer. Therefore, the accuracy of the different V_0 method is increased when the value of the maturity is higher, which is the same conclusion that was stated in the results section regarding the third order approximation.



Figure 19. Second Order Approximation and Heston Exact Solution at Different Values of κ .

In figure 19, the results show that the second order approximation yielded a better precision than the other methods. Although this one always bounded the exact solution from below, the results of the fair volatility strike price are almost equal. If one observes figure 19, there is no need for a higher value of kappa to obtain a result which is almost the same as the exact solution. However, one may still state the same conclusion in which higher values of kappa increase the accuracy of the approximation.



Figure 20. Second Order Approximation and Heston Exact Solution at Different Values of Maturity.

In figure 20, one may see that the pattern described by both approaches is the same as the one obtained from the other methods. As it was mentioned before, the second order approximation achieved a better convergence than the other methods. The reasons of the latter statement have already stated on the results section. Finally, as ne may recall from figure 17 to 20, the conclusions applied for the third order approximation, may also be applied for the two other methods, the different V_0 and the second order expansion.

Section C



Figure 21. VIX Index Implied Variance.

Figure 20 represents the shape of the implied variance of the VIX Index obtained from a calibration proportioned by Bloomberg. In particular, figure 20 represents a three months implied variance of the VIX Index, obtained on Monday 20 of August of 2012.

Documentos de Trabajo Banco Central de Chile	Working Papers Central Bank of Chile	
NÚMEROS ANTERIORES	PAST ISSUES	
La serie de Documentos de Trabajo en versión PDF puede obtenerse gratis en la dirección electrónica:	Working Papers in PDF format can be downloaded free of charge from:	
www.bcentral.cl/esp/estpub/estudios/dtbc.	www.bcentral.cl/eng/stdpub/studies/workingpaper.	
Existe la posibilidad de solicitar una copia impresa con un costo de Ch\$500 si es dentro de Chile y US\$12 si es fuera de Chile. Las solicitudes se pueden hacer por fax: +56 2 26702231 o a través del correo electrónico: <u>bcch@bcentral.cl</u> .	Printed versions can be ordered individually for US\$12 per copy (for order inside Chile the charge is Ch\$500.) Orders can be placed by fax: +56 2 26702231 or by email: <u>bcch@bcentral.cl</u> .	

DTBC-707

A PCA Approach to Common Risk Exposures in the Chilean Banking System Diego Avanzini y Alejandro Jara

DTBC - 706

Coordination of Expectations and the Informational Role of Policy

Yang Lu y Ernesto Pastén

DTBC - 705

Wicksell versus Taylor: A Quest for Determinacy and the (Ir)relevance of the Taylor Principle Sofía Bauducco y Rodrigo Caputo

DTBC-704

The Changing Roles of Education and Ability in Wage Determination Gonzalo Castex y Evgenia Dechter

DTBC - 703

Proceso de Endeudamiento y Sobre Endeudamiento de los Hogares en Chile Jaime Ruiz-Tagle, Leidy García y Álvaro Miranda

DTBC - 702

The Wealth Distribution in Developing Economies: Comparing the United States to Chile

Sofía Bauducco y Gonzalo Castex

DTBC – 701 Impacto de Sorpresas Macroeconómicas y Anuncios en Factores de la Estructura de Tasas de Chile Luis Ceballos

DTBC – 700 A Note on Yield Spread and Output Growth Rodrigo Alfaro y Damián Romero

DTBC - 699

Comportamiento de no Pago en Créditos de Consumo: Indicadores y Determinantes Andrés Alegría, Rodrigo Alfaro y Carlos Saavedra

DTBC - 698

A Simple Out-of-Sample Test for the Martingale Difference Hypothesis Pablo Pincheira

DTBC - 697

Vulnerability, Crises and Debt Maturity: Do IMF Interventions Shorten the Length of Borrowing? Diego Saravia

Diego Saravia

DTBC - 696

Are All Capital Inflows Associated with *Booms* in House Prices? An Empirical Evaluation

Alejandro Jara y Eduardo Olaberría

DTBC - 695

Household Debt During the Financial Crisis: Micro-Evidence from Chile Roberto Álvarez y Luis Opazo

DTBC – 694 Efficient CPI-Based Taylor Rules in Small Open Economies Rodrigo Caputo y Luis Oscar Herrera

DTBC – 693 Interventions and Inflation Expectations in an Inflation Targeting Economy Pablo Pincheira



BANCO CENTRAL DE CHILE

DOCUMENTOS DE TRABAJO • Diciembre • 2013