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THE IMPACT OF PERSISTENCE IN VOLATILITY OVER THE PROBABILITY OF DEFAULT*

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Abstract
We evaluate the impact of persistence in volatility over the probability of default in Merton's credit risk model. Our main conclusion is that a high degree of persistence, as it is observed in equity returns, implies a lower probability of default for those cases where firms possess a high level of leverage.

Resumen
En este artículo evaluamos el impacto de la persistencia en volatilidad en la probabilidad de no pago del modelo de riesgo de crédito de Merton. Nuestra conclusión principal es que una elevada persistencia, como la observada en los retornos accionarios, implica una menor probabilidad no pago en el caso de firmas con alto nivel de endeudamiento.

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I. Introduction

One of the main structural models for credit risk is the one proposed by Merton (1974). In that paper the market value of assets follows a geometric Brownian motion with constant volatility, and therefore a firm defaults when the market value of an asset is below a given debt threshold. A main drawback in Merton's model is the assumption of normality, which implies a probability of default that does not match actual defaults. A 'solution' for this issue is provided by Moody's through an estimate of the probability of default, called EDF (Expected Default Frequency). Alternatively, a more general default probability can be obtained by allowing the underlying process to have stochastic volatility. In that case the conditional distribution can be obtained by finite-difference methods using a Partial Differential Equation for valuating a binary call (also called Kolmogorov Backward Equation).

Instead of taking that approach, Fouqué et al. (2001) propose an asymptotic expansion of the distribution based on the parameter of mean reversion of the volatility process. Gerbasch and Surulescu (2010) apply that method for the case of Merton's model under the assumption of uncorrelated disturbances, meanwhile Fouqué et al. (2006) compute an asymptotic expansion for the yield spread obtained under Black and Cox's (1976) model. The main limitation of asymptotic analysis is that the approximation is valid only locally, which in this case includes only a volatility process with low degree of persistence (or high degree of mean-reversion).

Moreover, standard tools in Financial Econometrics for modeling equity returns are conditional volatility models (Campbell et al., 1997), for which the GARCH's family is the most popular approach (Andersen et al., 2010). The relationship between GARCH models and stochastic volatility models is that the continuous-time limit of the former is a particular case of the latter (Corradi, 2000; Singleton, 2006). Indeed, the continuous-time limit of GARCH model is nested in the Heston-Nandi model, a special case when disturbances of volatility equation and price equation are perfectly correlated (Gatheral, 2006). However, under Corradi's setup, it is not possible to distinguish between

GARCH(1,1) and ARCH(1) in continuous-time, thus we use an ARCH(1) model in our Monte Carlo experiments.

The main purpose of our paper is to quantify the impact of a high degree of persistence (in volatility process) on the probability of default, meaning that we introduce conditional volatility into Merton's model. In contrast with Gerbasch and Surulescu (2010), when asymptotic expansion is proposed, we use Monte Carlo experiments of an ARCH(1) model. We use simulations since we are interested in ARCH(1) models with high degree of persistence, which are poorly approximated by asymptotic expansions. Our main conclusion is that a high degree of persistence in the conditional volatility model —as it is usually observed in equity returns— does reduce the probability of default for firms with low credit quality (medium and high level of leverage).

The paper is organized as follows. Section II introduces the model, Section III shows simulation results for a selected set of parameters, and Section IV concludes.

II. Analytic Framework

In this section we discuss the Data Generating Process for the underlying process (asset value) and the procedure to estimate the Probability of Default (PD). The continuous-time counterpart of the model is discussed in order to compare our numerical results with previous literature on stochastic volatility (Fouqué et al., 2001; Fouqué et al., 2006; and Gerbasch and Surulescu, 2010).

1. Asset Value DGP

Let us suppose that the return on assets (r_t) can be modeled by a GARCH(1,1) model:

$$\begin{aligned} r_t &= \left(\mu - \frac{1}{2}h_t\right) + \sqrt{h_t}\,e_t \\ h_t &= \sigma^2(1-\alpha-\beta) + \alpha h_{t-1}e_{t-1}^2 + \beta h_{t-1} \end{aligned}, \qquad (1)$$

where h_t is conditional variance and e_t is a standard Gaussian disturbance (zero mean and unit variance). Also, σ, α, and β are non-negative; and $\alpha + \beta \leq 1$. In particular when $\alpha + \beta = 1$, the model is a integrated-GARCH or IGARCH(1,1)[1] in which case there is not an unconditional variance, although the process is still stationary (Campbell et al., 1997).[2]

In order to compare our results with stochastic volatility models, we adopt Corradi's (2000) continuous-time limit of GARCH model[3]. In particular, we consider that: (i) time-interval can be subdivided in Δ steps, and (ii) the rate of convergence of volatility is lower than Δ, which implies: $\lim_{\Delta \to 0} \alpha_\Delta / \Delta^\delta$ ($\delta < 1$).

Based on the second assumption we have:

$$\begin{aligned} h_{t\Delta}^2 &= \sigma_\Delta^2 (1 - \alpha_\Delta - \beta_\Delta) + \alpha_\Delta h_{(t-1)\Delta} e_{(t-1)\Delta}^2 + \beta h_{(t-1)\Delta} \\ &\cong \sigma_\Delta^2 (1 - \alpha_\Delta - \beta_\Delta) + (\alpha_\Delta + \beta_\Delta) h_{(t-1)\Delta} \end{aligned}.$$

Note that previous equation is a discrete-time version of the following deterministic differential equation:

$$dv(t) = \theta(\bar{v} - v(t))dt, \tag{2}$$

[1] Since the introduction of that model by JP Morgan in their risk-toolkit RiskMetrics in 1997, the IGARCH have been widely used by practitioners that take calibrated values for "beta".
[2] It should be noted that a difference between this model and the standard GARCH model (Campbell et al., 1997) is the convexity-adjustment term in the return equation, which implies a time-variant mean process. In continuous time, the term is obtained from the application of Ito's calculus to a geometrical Brownian motion process with time-variant volatility. Indeed, our model is a special case of Heston-Nandin model when disturbances of volatility equation and price equation are perfectly correlated (Gatheral, 2006).
[3] The limit of GARCH(1,1) model and its relationship with stochastic volatility model is still an open issue (Alexander and Lazar, 2005; Singleton, 2006). However, we consider Corradi's approach because her result keeps the condition that GARCH's model has only one source of randomness.

where $v(t)$ is the continuous-time limit of $h_t, \bar{v} \equiv \lim_{\Delta\to 0} \sigma_\Delta^2 / \Delta$ (long-run variance) and $\theta \equiv -\lim_{\Delta\to 0} \log(\alpha_\Delta + \beta_\Delta)/\Delta$ (persistence). Thus, a so called fast-reverting process is characterized by $\theta \to \infty$.

Several things must be considered at this time:

- The continuous-time limit is only valid for non-integrated GARCH models, which means $\alpha + \beta < 1$ and therefore $\theta > 0$. In practice, we do observe IGARCH models for equity returns, which could be approximated in our simulations by volatility process with a high-degree of persistence.

- When persistence is small, the process reverts very quickly to its long-run level, thus there is a small amount of uncertainty regarding the volatility measure.

- Under Corradi's setup, it is not possible to distinguish between GARCH(1,1) and ARCH(1) models. Therefore we present our results in terms of the persistence parameter which can be obtained from both models.

- Our expression for continuous-time persistence is consistent with an exact discrete-time Ornstein-Uhlenbeck process (Phillips and Yu, 2001). Given the previous discussion, we restrict our Monte Carlo experiments to stable ARCH(1) models, and therefore the parameters of the DGP are: $\sigma > 0$, and $0 \le \alpha < 1$.

2. Estimates of PD

Under Merton's model a closed-form expression for PD can be obtained. However under a general DGP for asset value a numerical solution must be used. In particular, we consider the cumulated returns over a fixed time period.[4] Thus, given L to be a fixed level of

[4] In contrast, Fouqué et al. (2006) examine the impact of stochastic volatility on PD and yield spreads under Black-Cox setup meaning that default can occur any time before maturity.

leverage (as percentage of the asset value at time t), a default occurs at time T, when the following condition is satisfied:

$$\sum_{t=1}^{T} r_t < \log(L). \tag{3}$$

Note that under constant volatility we have:

$$X_T \equiv \sum_{t=1}^{T} r_t = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sum_{t=1}^{T} e_t \ .$$

Last term is a sum of independent normal disturbances with unit variance; therefore it is distributed normal with variance equals to T.

Thus, PD is obtained as follows:

$$\begin{aligned} PD &\equiv \Pr\left[X_T < \log(L)\right] \\ &= \Pr\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sum_{t=1}^{T} e_t < \log(L)\right] \\ &= \Phi\left(\frac{\log(L) - \left(\mu - \sigma^2/2\right)T}{\sigma\sqrt{T}}\right) = \Phi(-DD) \end{aligned}$$

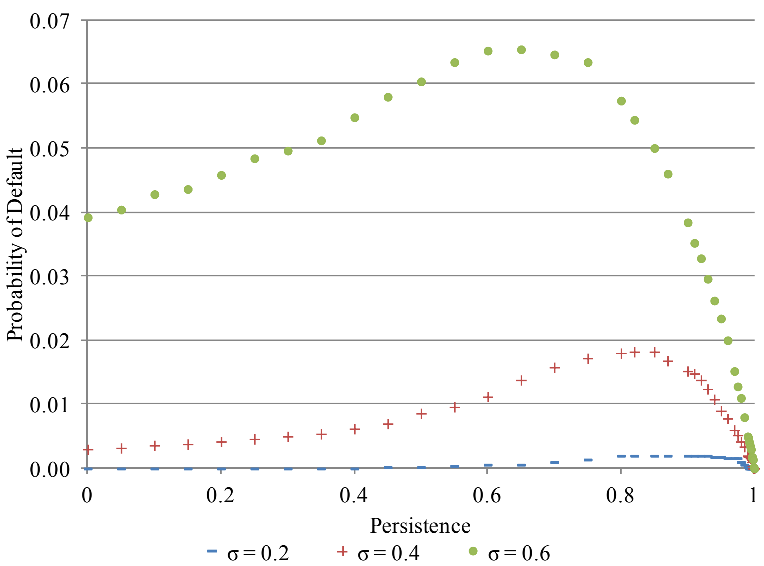
where DD stands for Distance to Default. DD is a widely used risk indicator, which is obtained combining both market and balance-sheet information (Gray and Malone, 2008).

III. Monte Carlo Experiments

In this section we use Monte Carlo experiments to assess the impact of persistence on the probability of default. For that we consider the model presented in (1), assuming for simplicity that μ is zero, and β is zero. The latter implies that return process is modeled as ARCH(1). Following condition (3), the PD is computed as the percentage of times that the cumulative returns (in weekly steps) are below the logarithm of the level of leverage (L) for a year of data (52 weeks). Results are based on 5000 replications.

For the case of low-level of leverage, we observe a hump-shape form between PD and the persistence parameter for all levels of annual volatilities (Figure 1)[5]. However, increasing persistence from 0.9 to 1 implies a reduction in PD even for the case with low volatility.

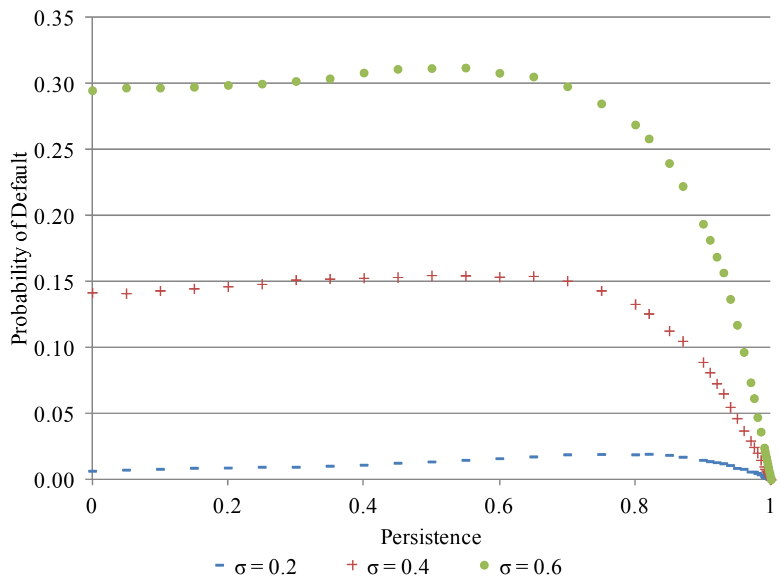
Figure 1: PD for a low-level of leverage (L = 30%)



Moving to a medium-level of leverage (60%) we observe almost no effect on PD when persistence parameter increases from zero to 0.7 (Figure 2). Also, there is a break-even level of persistence in which PD is decreasing on that parameter.

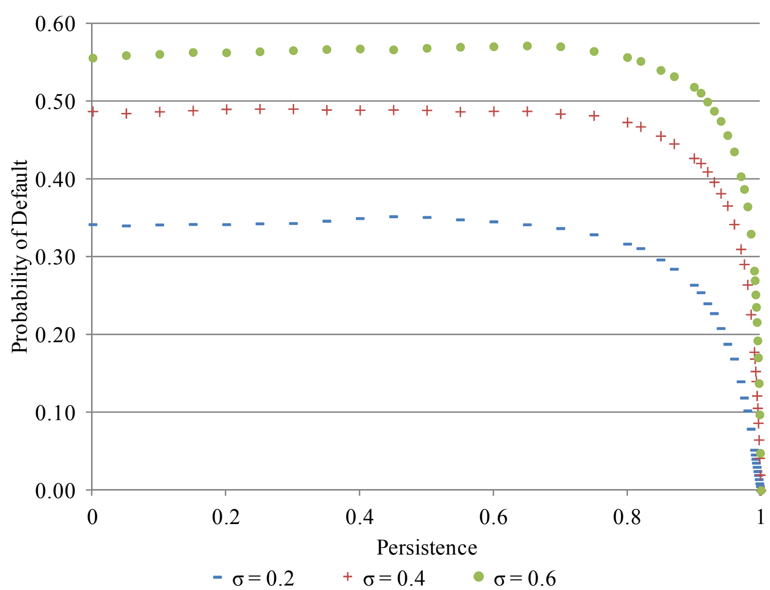
[5] Additional levels of volatilities were also considered leading similar qualitative conclusion. Details of those results are available from the authors upon request.

Figure 2: PD for a medium-level of leverage (L = 60%)



Finally, for high-level of leverage, we confirm our previous results, having an inverse L-shape form between PD and the persistence parameter (Figure 3).

Figure 3: PD for high-level of leverage (L = 90%)



IV. Conclusion

In this paper we introduce Conditional Volatility into Merton's credit risk model. Based on simulations of an ARCH(1) model, we conclude that persistence has a negative effect on the Probability of Default for the cases where firms have medium or high level of leverage (60 and 90% as percent of total asset, respectively). Although the conclusion may look counterintuitive, the finding was also stretched in Gerbasch and Surulescu (2010). A time-variant volatility can push away the value of asset and thus the so called Distance to Default (DD), by doing that it may reduce the Probability of Default.

A practical implication of this result is the computation of DD. Duffie and Wang's (1994) approach implies constant volatility, meanwhile Gray and Malone's (2008) two-equations-two-unknowns system is compatible with time-variant volatility, although the baseline model does not have that property. Thus, if the underlying process has conditional volatility, then Gray and Malone's approach provides a more accurate estimate of DD than Duffie and Wang's one. The difference between both approaches will increase as the degree of persistence in the volatility process is high.

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