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Alberto Naudon

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Agustinas 1180
Teléfono: (56-2) 6702475; Fax: (56-2) 6702231

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A STOCHASTIC ASSIGNMENT MODEL

Alberto Naudon
Gerencia de Análisis Macroeconómico
Banco Central de Chile

Abstract
In this article I study a dynamic stochastic extension of the "differential rents" model of Sattinger (1979) that generates endogenous mismatch in equilibrium. I depart from the standard assignment literature in assuming that agents' characteristics may change over time and that re-matching is feasible but costly. I show that, when agent characteristics are stochastic, rigidities that prevent partners from re-matching may change the matching outcome, even if the level of output continues to satisfy monotone differences in type. I construct and prove the existence of an equilibrium characterized by (i) a positive assortative matching between agents that decide to re-match and (ii) bands of inaction for existing matches.

Resumen
En este artículo se estudia una extensión dinámica estocástica del modelo de "rentas diferenciales" de Sattinger (1979) que genera desajustes en la forma de "emparejamiento" en equilibrio. El modelo se aparta de los modelos estándares de asignación de tipos en dos dimensiones: se permite que las características de los agentes cambien en el tiempo y se permite cambiar de parejas, aunque a cierto costo. Se muestra que, bajo estos dos supuestos, las rigideces que dificultan que los agentes cambien de pareja pueden cambiar el resultado del emparejamiento, incluso si el nivel de producto satisface diferencias monótonas en los tipos. Se construye y prueba la existencia de un equilibrio donde el emparejamiento es positivamente seleccionado entre los agentes que quieren cambiar de pareja y (ii) por bandas de inacción para las parejas existentes.

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Address: Banco Central de Chile, Agustinas 1180, Santiago, Chile; email: anaudon@bcentral.cl

1 Introduction

This article study a dynamic matching problem that generates endogenous mismatch in equilibrium. The model is a dynamic stochastic extension of the "differential rents" model of Sattinger (1979) where a continuum of two types of heterogeneous agents can produce only in pairs using a supermodular production function. I depart from the standard assignment literature in assuming that agents' characteristics may change over time. In the model re-matching is feasible but costly. The last element is important, since letting characteristics change over time with no cost of re-matching (i) would keep positive assortative matching (no mismatch in equilibrium) and (ii) would imply that agents change partners whenever a change in characteristics occurs regardless the size of the change, two features that seem unrealistic. I show that, when agent characteristics are stochastic, rigidities that prevent partners from re-matching may change the matching outcome, even if the level of output continues to satisfy monotone differences in type.

As in any typical assignment model, here both the assignment and the rewards functions are obtained simultaneously as the outcome of the interaction of two populations seeking partners in a competitive market. An important difference between this model and the standard static one is that, due to the cost of re-matching, not all agents are looking for partners. This implies that the distribution of characteristics across agents in the market is, in general, different from the (known) distribution of those characteristics across the whole population. This feature complicates the solution of the problem, since the assignment function can not be obtained in advance, as is common in typical models of this type.

I construct and prove the existence of an equilibrium characterized by (i) a positive assortative matching between agents that decide to re-match and (ii) bands of inaction for existing matches. In particular, for every type of agents there is set of partner characteristics that are acceptable in equilibrium, in the sense that they are not willing to pay the cost to re-match in order to match the "ideal" partner. I also provide an algorithm to compute the matching function and the inaction sets.

To the best of my knowledge, this is one of the first papers that tries to address assignment problems with time-varying random characteristics. One exception is Anderson and Smith (2008) that considers a model where agents gradually learn their types and show that positive assortative matching is, in most of the cases, not an equilibrium outcome even if the production function is supermodular. A similar idea appears in Papageorgiou (2007). This paper considers a search equilibrium market model where workers differ on their types (that remain unchanged throughout the course of their life). In particular, this paper considers a two types - two occupations model where different types have different productivities across occupations. Workers underlying productivity is revealed over time affecting worker's wage and occupational mobility decisions. In my model types are known but evolve over time. I show that in the model, supermodularity of the production function is sufficient for positive assortative matching among agents starting a new relationship.

The paper is related to a large literature that, starting with the seminal contributions of Roy (1951), Tinbergen (1951) and Koopmans and Beckmann (1957), have used static assignment models to address economic problems where indivisible characteristics of matched partners are fundamental[1]. For instance, these models has been widely used to study the distributions of earnings (see Sattinger (1993) for a review) and to explain the formation of production teams. Becker (1973) popularized this type of models to explain how people choose to marry and Garicano (2000) uses similar ideas to model hierarchies and the organization of knowledge. More recently, assignment models have been used to explain CEO earnings (Gabaix and Landier (2008) and Tervio (2008)) and to explain the increase in the level and dispersion of house prices across U.S. metropolitan areas (Van Nieuwerburgh and Weill (2006)). The list of applications is, of course, much longer, since situations where agent characteristics matter for the outcome of an association are varied and large.

In spite of the many useful insights from the static assignment model, it is easy to imagine situations where agents' characteristics change over time and push people to break their current matches. For instance, people change and a marriage that looked ideal when it was formed could become far from ideal some years later. In fact, in 2001, 13% of men and 14% of women had married twice, and around 3% had married three or more times (see Kreider (2005)). The same is true for a firm - CEO relationship, since a manager that was ideal under some firm conditions could become the wrong one when those conditions change. For instance, a number between 10% and 15% of CEO leave their position in a typical year (see Kaplan and Minton (2006) and reference therein). In the same vein, Holmes and Schmitz (1996) using data from 1982 *Characteristics of Business Owners* shows that transferred businesses as a percentage of all surviving businesses is around 5% if the business is managed by the founder, and between 16% and 27% otherwise, depending on the age of the business (see also Holmes and Schmitz (1995)).

But we see not only that incentives to remain paired change over time, but also that those non-ideal matches are not immediately destroyed despite there are better possibilities in the market. This means that, at least for a time, matches that are not ideal keep working together.

The importance of mismatch for total factor prductivity level has been recently emphasized in the developmentliterature. According to Restuccia and Rogerson (2008) how aggregate resources are allocated across uses is important for the understanding of cross-country differences in per capita incomes. Hsieh and Klenow (2007) make a similar claim after studying the case of manufacturing in China and India. In particular, according to these authors, if productive factors were reallocated to "equalize marginal products to the extent observed in the U.S." these countries would realize TFP gains of more that 30%. This paper differs from that literature in two respects. First, here I analyze indivisible factors misallocation rather than the misallocation of divisible factors, like capital and labor, and so an assignment problem arises. Indivisible production factors are critical for productivity. An example of an indivisible productive factor that have had large attention in the literature is managerial talent (see for example, Bloom and Reenen (2007)). Second, the misallocation of resources here is endogenous, in the sense that it arises from optimizing agents that consider the trade-off between gains and losses of being incorrectly matched. A tentative conclusion

[1]For a recent theoretical treatment of this kind of model, see Legros and Newman (2002) and Legros and Newman (2007). Assignment models are usually analyzed in competitive and frictionless market environments. For an alternative approach that considers search frictions see Shimer and Smith (2000).

from the numerical exercise presented at the end of the chapter is that it is difficult to find important aggregate effects of endogenous misallocation. The problem is that misallocation is important only when types are very different, but it is in these situations that incentives to tolerate mismatches are low, and a huge cost of rematching is needed to see large productivity effects.

The structure of the paper is as follows. Section 2 presents the model and shows that equilibrium exists. Section 3 constructs and proves the existence of an equilibrium characterized by (i) a positive assortative matching between agents that decide to re-match and (ii) bands of inaction for existing matches. Section 4 describes the algorithm used to find the equilibrium. Section 5 shows that the model could be interpreted as solving for per period payments rather than expected present value, an interpretation that relevant for some empirical applications. Section 6 further characterizes the equilibrium by analyzing the effects of various changes in model parameters on model outcomes. Section 7 extends the model in two different ways. First, introduces tenure effects and second allows for a more general stochastic process. Section 8 concludes. The main proofs are in the appendix.

2 The model

The model is an stochastic dynamic extension of the "differential rents" model of Sattinger (1979) that allows for time varying characteristics[2]. A cost of re-matching that generates endogenous mismatch in equilibrium is introduced. To be more concrete in the exposition of the model, I will name the sides of the match entrepreneurs and firms. Entrepreneurs are endowed with a *time invariant talent* to manage firms. Firms, in contrast, have a *time varying productivity*[3]. Entrepreneurs buy and sell firms in a competitive market.

Setup There is a continuum of entrepreneurs and firms which mass is normalized to 1. Entrepreneurs differ in their time invariant ability to manage a firm $a \in \mathcal{A} = [a_L, a_H] \in \mathbb{R}_+$, which is distributed across them according to a continuous probability distribution function $H(a)$ with density function $h(a)$. Firms, in turn, are characterized by a productivity level $b \in \mathcal{B} = [b_L, b_H] \in \mathbb{R}_+$, which, in a departure from the standard assignment models, is allowed to evolve over time. I make the following assumptions about stochastic process for firm productivity:

Assumption 1 *Given any current productivity b, with intensity λ the productivity level change to $b' \sim G(.)$, where $G(.)$ is a distribution function with support $\mathcal{B}$ and with density function $0 < p_g \leq g(.) \leq P_g < \infty$ for all b in $\mathcal{B}$.*

Assumption 2 *$0 < p_h \leq h(.) \leq P_h < \infty$ for all a in $\mathcal{A}$.*

[2]For a good description of Sattinger (1979) model (and for a complete survey on assignment models) see Sattinger (1993). The model has been used recently by Tervio (2008) and Gabaix and Landier (2008) to explain the evolution of CEO earnings.

[3]The model could be extended to have stochastic types on both sides. However, for expositional purposes I consider time variability in only one side. In the appendix I explain how the model works when both sides have stochastic types.

Here λ is a measure of shock persistence. In particular, a *higher* value of λ implies a *less* persistent shock, since the intensity at which productivity changes is higher when λ is higher. This specification was used, for example, by Asplund and Nocke (2006) in their investigation of the effect of fixed costs and market size on entry and exit rates and the age distribution of firms. The critical implication of assumption 1 is that whenever a firm is hit by a shock, the new productivity level is independent of the old one[4].

The production unit in the economy is an entrepreneur - firm matched pair (a, b). Each pair produces an amount determined by the production function $f(a, b)$, which is assumed to be continuous, continuously differentiable, bounded on the support of a and b and strictly increasing in both of its arguments. Both characteristics, a and b, are complements in production. More formally,

Assumption 3 (Strict Super Modularity) *The production function* $f(a, b)$ *is strictly supermodular. That is, for* $a^1 > a^2$ *and* $b^1 > b^2$*,* $f(a^1, b^1) + f(a^2, b^2) > f(a^1, b^2) + f(a^2, b^1)$*. Equivalently, since* $f(a, b)$ *is differentiable,* $f_{ab}(a, b) > 0$*.*

At any point in time, entrepreneurs can sell and buy firms in a competitive and frictionless market (i.e. there is no search or any other friction).[5] However, every time a new firm is bought, the entrepreneur has to pay a fixed and sunk cost $c \geq 0$. To avoid complications due to entry and exit, I assume that both outside options of entrepreneurs and firms scrap value are low enough such that all types participate in this market. Finally, for technical reasons that will become apparent later, I assume that with intensity $\eta > 0$ matches are broken. This value could be as close to zero as desired, so in any quantitative application its effect is negligible. I analyze the steady state of this economy.

Entrepreneur problem Entrepreneurs face two problems. First, they must decide between keeping or changing the firm they currently have. Second, if they decide to change the firm, they must choose which type of firm to buy.

Starting with the second problem, let $V(b)$ denotes the price of a firm with current productivity b and $M(a, b)$ the value of a match between a firm with productivity b and an entrepreneur with talent a. The problem of an entrepreneur with talent a that is looking for a firm to buy is then

$$\max_x \{M(a, x) - V(x) - c\}. \tag{1}$$

That is, when buying a firm, entrepreneurs choose the firm that maximizes the difference between the value of a particular firm under their property, $M(a, x)$, and the cost of such a firm $V(x)$ minus starting cost c.

[4]This property substantially simplifies the characterization of the equilibrium, since it helps to forward super modularity from the production function to the value functions, which is not necessarily true for more complex stochastic process. However, there are some unsatisfactory consequences of assumption 1. In particular, since new $b's$ are independent of old ones, any "learning" type interpretation of the stochastic process is ruled out. In section 7 I generalize the process assumed in 1 by letting the new productivity of the firm be a function of current entrepreneur ability.

[5]The model could be easily reinterpreted to accommodate an alternative market structure where firms with time-varying productivity hire managers.

Let $W(a) := \max_x \{M(a,x) - V(x) - c\}$ denote the output of this maximization problem. Then the gain of changing firm for a type a entrepreneur currently producing with a type b firm is $W(a)+V(b)$, since she may always sell the firm at $V(b)$ and start again buying the firm she prefers and obtaining $W(a)$. Therefore, the problem of a type a entrepreneur that owns a firm with current productivity b is described by the following Hamilton-Jacobi-Bellman (HJB) equation,

$$rM(a,b) = \begin{array}{c} f(a,b) + \lambda \left\{ \int \max \begin{bmatrix} M(a,b'), \\ W(a) + V(b') \end{bmatrix} G(db') - M(a,b) \right\} \\ +\eta \{W(a) + V(b) - M(a,b)\}. \end{array} \tag{2}$$

That is, for a type a entrepreneur the flow value $rM(a,b)$ of having a type b firm is equal to the production flow $f(a,b)$ plus the expected capital gain or loss that occurs either when productivity changes or when the match is exogenously destroyed. In particular, with intensity λ productivity of the firm changes from b to b' and in this event the entrepreneur can either keep the firm with a capital gain/loss of $M(a,b') - M(a,b)$, or sell it at a price $V(b')$. In the latter case she would get $W(a)$, with a capital gain equal to $W(a)+V(b')-M(a,b)$. Of course, the sale will occur only if the value of doing so is greater than the value of keeping the firm with the new productivity b'. Finally, in the case of an exogenous destruction, capital gains/losses are equal to $W(a)+V(b)-M(a,b)$.

The solution to entrepreneur problem can be characterized by two objects. A set $\mathbf{B}(a) \in \mathcal{B}$ of firm productivities that are acceptable for a entrepreneur with talent a and a firm productivity level $\beta(a) \in \mathbf{B}(a)$ that an entrepreneur a would like to buy in case she decide to change firms[6]. An important property of the value of a match (a,b) is the following.

Proposition 4 *$M(a,b)$ is supermodular.*
Proof. *It follows directly from equation (2), since the cross derivative is $\frac{f_{ab}(a,b)}{r+\lambda+\eta} > 0$ by assumption 3.* ■

The market for firms The solution to entrepreneur's problem induce a mass of firms of different types that are for sale and a mass of entrepreneurs of different talents that are looking for new firms. Let $A \in \mathcal{A}$ be a subset of entrepreneur types and $D(A)$ the mass of managers with talent $a \in A$ that are looking for a new firm. In the same vein, let $B \in \mathcal{B}$ be a subset of firm types and $S(B)$ the mass of firms with productivity $b \in B$ that are for sale. Note that $\beta(A)$ is the subset of firm types chosen by entrepreneurs in A. The market for firms will be in equilibrium when $D(A) = S(\beta(A))$ for all $A \subseteq \mathcal{A}$. That is, when the mass of entrepreneurs looking for a particular subset of firm types is equal to the mass of this subset of firm types in the market.

Equilibrium An equilibrium in this economy is as follows,

[6]So far, there is nothing proving that there is only one firm type that solves equation (1), but I show below that this will be the case, so, I treat $\beta(a)$ as a function.

Definition 5 *An equilibrium in this economy is characterized by a price function $V : \mathcal{B} \to \mathbb{R}$; an assignment function $\beta : \mathcal{A} \to \mathcal{B}$; and inaction sets $\mathbf{B}(a)$ for all a, such that given the payment function $V(b)$, the assignment function $\beta(a)$ and the inaction set $\mathbf{B}(a)$ (i) solve entrepreneur problem and (ii) $S(A) = D(\beta(A))$ for all $A \subseteq \mathcal{A}$.*

3 Equilibrium construction

In this section I construct and prove the existence of an equilibrium characterized by (i) a positive assortative matching between agents that decide to re-match and (ii) bands of inaction for existing matches.

Positive assortative matching in the market for firms means that the equilibrium assignment is given by a bijective function, that with some abuse of notation, I denote by $\beta(a)$, with $\beta(a_L) = b_L$, $\beta(a_H) = b_H$ and $\beta'(a) > 0$ for all $a \in \mathcal{A}$.

As in any assignment model, both the assignment problem and the price functions are solved simultaneously. However, there is a critical difference between the dynamic version of the model presented here and the standard static one that has been used in the literature. Here the assignment problem is between populations of firms and entrepreneurs that, endogenously, decide to go to the market, and not between the whole population of both types of managers and firms. The later implies that, even with positive assortative matching in the market for firms, I cannot use the known distribution of talent and productivity to infer the assignment function. Here the assignment function is the solution to a fixed point problem.

For expositional purposes, I will describe the payment functions first and I will postpone the discussion of the assignment function to section 3.3. In that section I show that, under assumptions (1) and (3), in equilibrium there exists a increasing bijective function $\beta : \mathcal{A} \to \mathcal{B}$ that specifies the type of firm $\beta(a)$ that is bought by an entrepreneur with talent a.

3.1 Equilibrium firm price, $V(b)$

Consider an entrepreneur that is looking for a new firm. In particular, think about the entrepreneur who chooses optimally to buy a type b firm and denote her talent by $\alpha(b)$. Clearly, $\alpha(b)$ is the inverse function of $\beta(a)$. Optimality implies that for any $\varepsilon > 0$ it has to be the case that $W(\alpha(b)) \geq M(\alpha(b), b + \varepsilon) - V(b + \varepsilon) - c$. That is, an entrepreneur of talent $\alpha(b)$ that is looking for a new firm prefers to buy a firm b to any other one. Re-writing this expression in terms of the value of the match, $M(\alpha(b), b)$, in equilibrium the following condition must be true,

$$M(\alpha(b), b) - V(b) \geq M(\alpha(b), b + \varepsilon) - V(b + \varepsilon). \tag{3}$$

Dividing equation (3) by ε and letting $\varepsilon \to 0$ I get a condition for the change in the value of the firm due to a (infinitesimal) change in productivity,

$$V'(b) = M_b(\alpha(b), b). \tag{4}$$

That is, in equilibrium, the marginal value of an improvement in productivity $V'(b)$ is equal to the marginal value for the equilibrium match, $M_b(\alpha(b), b)$. From equation (2) it is clear that, for any (a, b), $M_b(a, b) = \frac{f_b(a,b)}{r+\lambda+\eta}$.

Equation (4) only determines the slope of the firm price function $V(b)$. In order to get an equation for the price, it is necessary to specify boundary conditions. I have assumed that both entrepreneurs and firms have a very low value outside the market, so all of them decide to participate. In order to find the boundary conditions, consider the value of the match between the worst entrepreneur and the worst firm, $M(a_L, b_L)$, and assume that it is divided between both sides in proportions $0 < \delta < 1$ for the entrepreneur and $1 - \delta$ for the firm. Clearly, $(1 - \delta) M(a_L, b_L)$ is a lower bound for firm price, since any firm in the economy has an equal or greater productivity level b. Then $M(a_L, b) \geq M(a_L, b_L)$ for all b. Same argument implies that $\delta M(a_L, b_L)$ is the lower bound for entrepreneur value. Any division will be an equilibrium since both sides strictly prefer to participate in the market. As is common in the literature, this problem could be rationalized as a Nash bargaining problem with δ being the entrepreneur's bargain power. In what follows I assume $\delta = 1$, so the value of the worst firm is set to 0. Then, assortative matching in firm market implies that the price of a type b firm is given by

$$V(b) = \int_{b_L}^{b} \frac{f_b(\alpha(x), x)}{r + \lambda + \eta} dx. \tag{5}$$

Note that, because $V'(b) = \frac{f_b(\alpha(b),b)}{r+\lambda+\eta}$, and in contrast to models where factors are fully divisible (i.e. models without assignment problem), changes in the value $V(b)$ due to improvements in productivity depend on both the marginal product of b and the entrepreneur that would be "assigned" in the market to manage a firm of productivity b. That is, a small increase in productivity could lead to a large increase in firm value if the talent of the entrepreneur that will manage the firm improves a lot. In general, it is difficult to obtain a closed form solution for $\alpha(b)$ and so equation (5) must be solved numerically[7].

The next proposition establishes that under this payment function entrepreneurs does not want to deviate from the assignment $\beta(a)$.

Proposition 6 *Assume that prices are set according to equation 5, then the assignment function $\beta(a)$ coincide with entrepreneurs optimal choice.*
Proof. *I derived the price function $V(b)$ assuming that the assignment function $\beta(a)$ was equal to entrepreneurs optimal choice. To see that this is actually the case, let $W(a|b) = M(a, b) - V(b) - c$ denote the value of a type a entrepreneur that bought a firm of productivity b. The optimality of the assignment $\beta(a)$ follows from the fact that, assuming (3) and (1), this function is strictly concave with respect to b and has a unique maximum at $\beta(a)$ for all $a \in \mathcal{A}$. To see this, note that the first derivative of $W(a|b)$ with respect to b is $M_b(a, b) - M_b(\alpha(b), b)$, and that by supermodularity of $M(.)$: (i) if $b = \beta(a)$ then this derivative is 0, (ii) if $b > \beta(a)$ then $\alpha(b) > a$ and this derivative is negative, and (iii) if $b < \beta(a)$ then $\alpha(b) < a$ and this derivative is positive.* ■

[7] In section 4 I present the algorithm used to compute the assignment function.

3.2 Inaction sets and separation policy

Now I turn to the separation policy. A direct implication of proposition (6) is that there is a simple rule to break up a match. To find this rule note that the set $\mathbf{B}(a)$ of acceptable firm types for an entrepreneur with ability a is determined as follows,

$$\mathbf{B}(a) := \{b \in \mathcal{B} : W(a|b) \geq W(a) - c\},$$

and note that since proposition (6) implies that $W(a|b)$ is strictly concave with a maximum at $\beta(a)$ equal to $W(a)$, this set is just an interval of $\mathbb{R}_+$ with boundaries $\left\{\underline{b}(a), \overline{b}(a)\right\} \subseteq \mathcal{B}$ defined by the solution to $W(a|b) = W(a) - c$. More precisely, proposition (6) implies that there are at most two solutions to this equation in $\mathcal{B}$, one greater than $\beta(a)$ and the other smaller. In the case where there is no solution for the lower bound (upper bound) in $\mathcal{B}$, the lower (upper) boundary is just b_L (b_H). This interval defines an inaction band such that the best response to a change in productivity that retains the firm inside the interval is to keep the firm. Therefore, the separation policy is: sell the firm whenever productivity shock b' lies outside the interval $\left(\underline{b}(a), \overline{b}(a)\right)$ and keep the firm otherwise. It is plain that the functions $\underline{b}(a)$ and $\overline{b}(a)$ defining the boundaries of the inaction band are non decreasing and continuous functions of a. (See figure 2)

For fixed b, it is possible to obtain the set of entrepreneurs' talents that in equilibrium may be working with a type b firm. This set is an interval of $\mathbb{R}_+$ with boundaries $\{\underline{a}(b), \overline{a}(b)\} \subseteq \mathcal{A}$ defined by,

$$\mathbf{A}(b) := \{a \in \mathcal{A} : W(a|b) \geq W(a) - c\}.$$

3.3 The assignment function and the market clearing condition

So far, I have assumed that the assignment function $\beta(a)$, indicating the type of firm bought by an entrepreneur a, exists and could be inverted to obtain $\alpha(b)$. In this section I show that this is indeed the case and I characterize this function. In order to do that, it is necessary to specify the supply and demand for firms described in section 2.

Let $\varpi(a) = \lambda \Pr[b \notin \mathbf{B}(a)]$ denotes the intensity at which entrepreneurs type a get a shock bad enough to make them to look for a new firm. Then $\varpi(a) + \eta$ is the intensity at which type a entrepreneurs sell their firms and the mass of entrepreneurs with talent less than a that are looking for a new firm is just

$$D(a) = \int_{a_L}^{a} h(x)\left[\varpi(x) + \eta\right] dx. \tag{6}$$

To compute the supply of firms with productivity less than b consider the set $\mathbf{A}(b)$ of entrepreneurs that would accept to work with firms with productivity b, and note that the law of large number implies that the mass of entrepreneurs that will never buy a firm b is given by $\omega(b) = \Pr[a \notin \mathbf{A}(b)]$. Recall that at a rate λ a firm own by entrepreneurs in the complement of $\mathbf{A}(b)$ is hit by a productivity shock from the distribution $G(.)$, therefore $\lambda g(b)\omega(b)$ is the measure of type b firm that are sold by those entrepreneur. Then, the supply of firms with productivity level less than b is given by,

$$S(b) = \lambda \int_{b_L}^{b} g(x) [\omega(x) + \eta] \, dx. \tag{7}$$

Of course, the total mass of entrepreneurs and firms in the market are the same in equilibrium, since every broken match means that one entrepreneur and one firm are going to the market. Let $\mu = D(b_H) = S(a_H)$ be this mass, then $S(a)/\mu$ and $D(b)/\mu$ are the distribution functions of talent and productivity in the market.

The market clearing condition is derive as follows. Supermodularity of the match value function implies positive assortative matching in the market for firms, that is, the worst available firm is bought by the worst entrepreneur that is looking for a firm and so on. Then, entrepreneurs of talent $\alpha(b)$ or less are working for firms with productivity b or less. The market clearing condition may be written as follows,

$$D(\alpha(b)) = S(b), \quad \textit{all } b \in \mathcal{B}. \tag{8}$$

The equilibrium assignment function is the one that simultaneously solves equation (8) and satisfies the value functions in section 3.1. To prove the existence of the function $\alpha(b)$ note that there are two maps describing the equilibrium in the model: one from assignment functions to inaction sets (discussed in section 3.1) and the other from inaction sets to assignment functions (the market clearing condition in this section). These two maps define the fixed point problem in equation (8). The next proposition establishes that there is an assignment function that solves the fixed point problem in equation (8).

Proposition 7 *Assume (1) and (3), there exists an increasing continuous bijective function $\alpha : \mathcal{B} \rightarrow \mathcal{A}$ that solves equation (8).*
Proof. *See appendix B.* ■

Proposition 8 *A positive assortative equilibrium exists.*
Proof. *By propositions 6 and 7 there exists a positive assortative assignment and associated payment function such that market clears and nobody wants to deviate.* ■

4 Algorithm

As was pointed out above, in general, it is not possible to get an analytical solution to the assignment function $\alpha(b)$. In this section I show the algorithm used to compute the equilibrium. The idea is basically to find a fixed point of an operator that maps the space of increasing continuous bijective functions that go from $\mathcal{B} \rightarrow \mathcal{A}$ into itself. To construct this operator I start by differentiating the equilibrium condition in equation (8),

$$\alpha'(b) = \frac{g(b)(\omega(b) + \eta)}{h(\alpha(b))(\varpi(\alpha(b)) + \eta)}. \tag{9}$$

Let $H(x,y) = \frac{g(x)(\omega(x)+\eta)}{h(y)(\varpi(y)+\eta)}$ and note that positive assortative matching implies $\alpha(b_L) = a_L$. Then, integrating equation (9), it is easy to see that the assignment function $\alpha(b)$ is the fixed point of the following functional equation:

$$\alpha(b) = a_L + \int_{b_L}^{b} H(x, \alpha(x))\, dx. \tag{10}$$

In appendix B I show that the solution to this problem is given by the fixed point of the following mapping,

$$T(\alpha)(b) := a_0 + \int_{b_0}^{b} \widehat{H}(x, \alpha(x))\, dx, \tag{11}$$

where $\widehat{H}(x,y) = \frac{a_H - a_L}{\varkappa(\alpha)} H(x,y)$ and $\varkappa(\alpha) = \int_{b_L}^{b_H} H(x, \alpha(x))\, dx$.

5 Per period payments

So far, I have set up the problem in terms of present values, but for some quantitative applications it is convenient to use per period flow payments. For example, if the model is applied to manager compensation and firm value questions as in Gabaix and Landier (2008) and Tervio (2008), one could replace the entrepreneur by a manager and think about a firm hiring a manger instead of a entrepreneur buying a firm. In this case one would be interested in the flow payment of the manager instead of the expected present value of those payments. Fortunately, the model could be solved under both interpretations [8]. Let $\phi(b)$ be the firm per period (or rental value of a firm type b) and let $w(a|b)$ be the per period payment received by a talent a entrepreneur managing a type b firm. To compute $\phi(b)$ note that the price of a type b firm has to satisfies the following HJB equation

$$rV(b) = \phi(b) + (\lambda + \eta)\, \mathbf{E}\{V(b') - V(b)\}. \tag{12}$$

Here the function $V(b)$ is just the firm price found in equation (5).

Entrepreneur per period flow $w(a|b)$ could be obtained using the fact that $f(a,b) = w(a|b) + \phi(b)$. Again, it is not possible to get closed form solutions in general, but several aspects of those functions could be stated. First, note that the slope of the firm per period flow is $\phi'(b) = f_b(\alpha(b), b)$. As with function $V(b)$, the slope of the per period flow payment depends both on the marginal product of b and the entrepreneur that manages the firm in equilibrium, $\alpha(b)$. The concavity or convexity of the payment function $\phi(b)$ will depend on two forces. To see this, note that differentiating $\phi(b)$ twice, the second derivative of firm per period payment flow is given by $\phi''(b) = [f_{b,b}(\alpha(b), b) + f_{a,b}(\alpha(b), b)\, \alpha'(b)]$. The first force is the behavior of the marginal product of b, $f_{b,b}(\alpha(b), b)$. The second depends on the complementarity and the change in the type of entrepreneur that would be optimally assigned to the firm in equilibrium, $f_{a,b}(\alpha(b), b)\, \alpha'(b)$.

With respect to entrepreneur per period flow, the condition $f(a,b) = w(a|b) + \phi(b)$ implies that the relation between entrepreneur payment and changes in firm productivity is

[8] Of course, this not the only per period payment structure consistent with the equilibrium.

governed by $w_b(a|b) = f_b(a|b) - f_b(\alpha(b), b)$. That is, it depends on the difference between the effect of productivity change on production in the firm with the current entrepreneur and in a firm managed by the optimal entrepreneur. Of course, if $a = \alpha(b)$ the wage is in its maximum level and so $w_b(a|b) = 0$. Whenever $a > \alpha(b)$, supermodularity of the production function implies that $w_b(a, b) > 0$ and then wages increase with b. In the same vein, $a < \alpha(b)$ implies that $w_b(a|b) < 0$ and then wages decrease with b. Then, for any given a the wage function is a concave function of b with a unique maximum at $\alpha(b)$.

Discussion Let us consider entrepreneur payment $w(a|b)$. First, note that conditional on a there is a distribution of payments $w(a|b)$. Indeed, as was pointed out above, $w(a|b)$ is equal to $f(a, b) - \phi(b)$, and because in equilibrium entrepreneurs of the same type a are working for firms with different productivity b, it has to be the case that entrepreneurs with the same talent get different payments. This in contrast with standard assignment models were conditional on factors characteristic there is a unique equilibrium payment.

How $w(a|b)$ changes with a or b? Let start fixing a and consider the effects a small moves in productivity. In the model, the relation between firm productivity and the wage of a type a entrepreneur is non monotonic. Actually, the non monotonicity between firms productivity and entrepreneur payment is at the core of assignment models, where, in equilibrium, it is not in the best interest of one side to be matched with the best of the other side, but every one has a preferred type to be matched with. To see this, note that the maximum possible payment for an entrepreneur of talent a occurs when she is paired with a firm of productivity $\beta(a)$. If with the passage of time the productivity of the firm decreases, both the production of the firm $f(a, b)$ and the firm payment $\phi(b)$ decrease, but supermodularity implies that, given a, $f(a, b)$ decreases more than $\phi(b)$ and therefore entrepreneur per period payment decreases too. Eventually, if the productivity of the firm decreases too much, the match will be destroyed. A similar process occurs when the productivity of the firm increases. In this case $f(a, b)$ increases less than $\phi(b)$ and therefore the entrepreneur per period payment decreases. Then an increase in the productivity of the firm will generate an increase in the payment of the entrepreneurs only when the firm is getting closer to the "optimal" firm for a given talent of the entrepreneur from below (i.e. from a productivity level smaller than $\beta(a)$). On the other hand, an increase in productivity will be associated with a decrease in entrepreneur payment if the improvement in productivity moves the firm above her "optimal" firm type. In order to have a monotonic positive correlation between firm productivity and entrepreneur payment it would be necessary to introduce some match-specific component with no value outside the firm, as in Jovanovic (1979). In section 7 I go over this issue, extending the model to allow for tenure effects in the match production function.

Now, let set b to see how $w(a|b)$ change with a. In this case the relation is always positive: an increase in a is always related to an increase in $w(a|b)$. The intuition here is much more natural: entrepreneurs are compensated only for relative performance. For a fixed a level of productivity b, the typical firm will produce $f(\alpha(b), b)$ and the firm will require a payment $\phi(b)$. Those managers working for firms of productivity b with talent greater than $\alpha(b)$ will get more than $w(\alpha(b)|b)$, since they have to make the same payment $\phi(b)$ to the firm, and those with less talent will get less. Then, conditional on firm type, firms that are doing better pay more than firms that are doing worse than the average.

6 Comparative statics

In this section I further characterize the equilibrium by studying the interaction between a change in model parameters and the shape of the inaction bands, the assignment function and the payment function. I also explore the effects of changes in the cost of re-matching and the persistence of the shock on the distribution of factor payments. I compare the results with ones obtained in a static model.

6.1 Inaction bands and turnover

Let me start by analyzing the behavior of the inaction bands and turnover. First, remember that for a given a the boundaries of the inaction set $\mathbf{B}(a)$ are the solution to $W(a|b) - W(a) = -c$ (of course, if there is no solution to this equation for $b \in \mathcal{B}$, the values of the boundaries are equal to the the upper or lower bound of $\mathcal{B}$ for the upper and lower boundary respectively). Figure 1 shows the value of $W(a|b) - W(a)$ as a function of b for a given value of a. As was stated above this function is concave with a unique maximum at $\beta(a)$. It is clear from panel (i) in figure 1 that as c increases so does the inaction band, since the intersection points with the line representing the cost of re-matching move away. A similar effect has an increase in λ, as is shown in panel (ii) of the same figure. An increase in λ means that the shock has less persistence. Then the cost of the mismatch is smaller in present value. For a given cost of re-match, there is more tolerance for matching with the wrong partner and so the inaction band increases.

How does the shape of the inaction sets change with differently shaped production functions? Production functions matter because the marginal product of each factor is relevant for entrepreneur decisions. However, as was mentioned above, what really matters in assignment models is the combination of marginal product of the factor in a traditional sense and the relative scarcity of the factor characteristic that, in turn, depends on the distribution of each characteristic. In order not to confuse these two elements, let me assume for now that both the productivity and the ability of the entrepreneurs come from a uniform distribution. In this case the effect of changes in the production function could be analyzed in isolation. Figure 2 shows the inaction band for two different production functions. Panel (i) of figure 2 shows the inaction band for the same production function of figure 1, $f(a,b) = ab$ and panel (ii) of figure 2 does the same for the production function $f(a,b) = \frac{a}{1-b}$. In the first case the marginal product of a is proportional to b and so the width of the inaction band is constant (for interior points). In contrast, in panel (ii) of figure 2 the marginal product increases more than proportional to b. In this case the inaction band is smaller for higher a, since for them the cost of moving away from the optimal level of b is higher. This example also helps to understand the effect of a change in the distribution of factor characteristics. To see this, note that having a production function $\frac{a}{1-b}$ where both a and b are distributed uniformly $[0,1]$, is equivalent to having a production function ab with b following a Pareto distribution with parameter 1 and support $[0,\infty)$. Then, panel (ii) in figure 2 shows that as

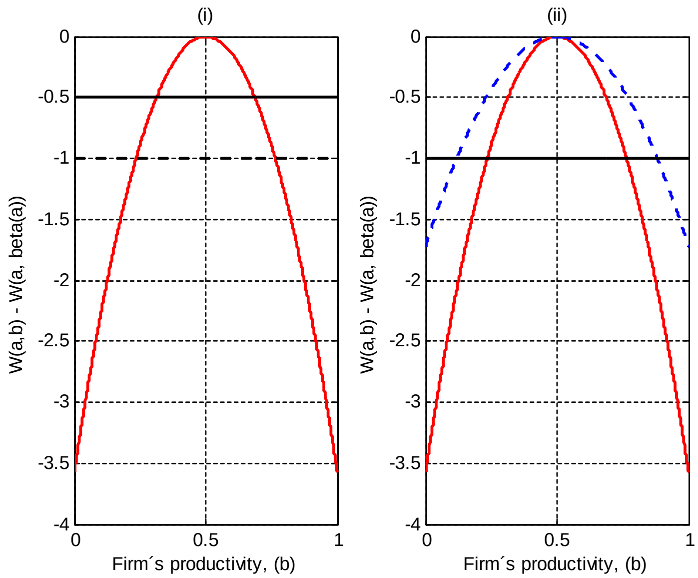

Figure 1: **Policy rule.** These figures depict the function $\Omega(a,b) = W(a,b) - W(a,\beta(a))$ for $a = 0.5$, $f(a,b) = ab$ and uniform distributions functions $H(.)$ and $G(.)$ with support $[0,1]$. In part (i) $\lambda = 0.1$. In part (ii) λ is set at 0.1 in the continuous red line and at 0.25 in the dashed blue line. The horizontal black lines represent the re-matching cost. The cut off values $\underline{b}(a)$ and $\overline{b}(a)$ (i.e., the boundaries of the inaction set) are the values of b for which the function $\Omega(a,b)$ intersects the value of the re-matching cost c.

higher productivity becomes relatively more scarce the inaction band becomes thinner.

Finally, the behavior of manager turnover mimics the behavior of the inaction band. In particular, thinner inaction bands imply more turnover.

6.2 Assignment Function

An obvious question is if the presence of a fixed cost of re-matching changes the shape of the assignment function relative to the case where the cost does not exist. The answer is, in general, yes.

As is clear from equation (9), in the dynamic assignment model, the assignment function depends critically on the functions $\varpi(a) = \lambda \Pr[b \notin \mathbf{B}(a)]$ and $\omega(b) = \lambda \Pr[a \notin \mathbf{A}(b)]$. That is, the function depends on the probability that the new productivity shock is outside the entrepreneur's inaction set and the probability that new inaction set does not contain the current entrepreneur ability (times the intensity of the shock).

To analyze the effects of changes in c let assume that $G(.)$ and $H(.)$ are uniform distribution functions. In this case, equation (9) becomes $\frac{\omega(x)+\eta}{\varpi(x)+\eta}$ and then the assignment function is just $\alpha(b) = b$ when $c = 0$ since in this case $\omega(.) = \varpi(.) = \lambda$. However, when $c \neq 0$ this ratio is, in general, different from 1, and so the shape of the assignment function is different from the case when $c = 0$, as the following example shows.

Example 9 *Consider $f(a,b) = a^{\theta_a} b^{\theta_b}$, and let $H(.)$ and $G(.)$ be uniform distribution functions with support in $[0,1]$. Assume that, as in the case with $c = 0$, $\alpha(b) = b$ regardless the*

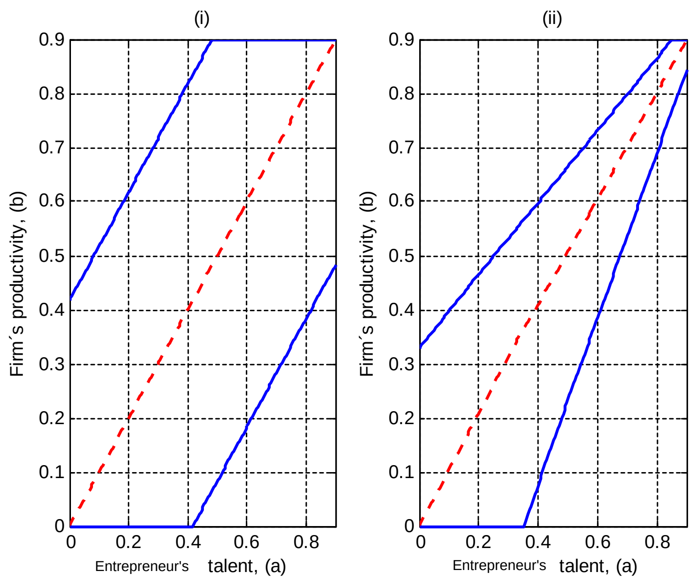

Figure 2: **Inaction bands.**
These figures depict the boundaries of the inaction set $\mathbf{B}(a)$ for all values of talent a, $\lambda = 0.25$, $c = 0.3$ and uniform distributions functions $H(.)$ and $G(.)$ with support $[0, 0.9]$. The production function is $f(a, b) = ab$ in part (i) and $f(a, b) = \frac{a}{1-b}$ in part (ii). The continuous red lines are the upper and lower boundaries and the dashed blue line is the assignment function $\beta(a)$. The picture in part (ii) is also the picture for a production function $f(a, b) = ab$ with $H(.)$ uniform with support $[0, 0.9]$ and $G(.)$ Pareto with parameter 1.

value for c. In this example I show that if $c > 0$ this could not be the equilibrium assignment function. First note that in this case the boundaries of the inaction set solve (for interior points),

$$\frac{1}{r+\lambda}\left\{a^{\theta_a}b^{\theta_b}-\frac{\theta_b}{\theta_a+\theta_b}b^{\theta_a+\theta_b}-a^{\theta_a+\theta_b}+\frac{\theta_b}{\theta_a+\theta_b}a^{\theta_a+\theta_b}\right\}=-c.$$

Now consider the boundaries at $a = 0$ and $b = 0$. It is clear that $\underline{a}(0) = \underline{b}(0) = 0$, $\overline{b}(0) = \left[\frac{\theta_a+\theta_b}{\theta_b}c(r+\lambda)\right]^{\frac{1}{\theta_a+\theta_b}}$ and $\overline{a}(0) = \left[\frac{\theta_a+\theta_b}{\theta_a}(r+\lambda)c\right]^{\frac{1}{\theta_a+\theta_b}}$, then

$$\alpha'(0)=\frac{\omega(0,0)}{\varpi(0,0)}=\frac{1-\left[\frac{\theta_a+\theta_b}{\theta_a}(r+\lambda)c\right]^{\frac{1}{\theta_a+\theta_b}}+\eta}{1-\left[\frac{\theta_a+\theta_b}{\theta_b}(r+\lambda)c\right]^{\frac{1}{\theta_a+\theta_b}}+\eta}.$$

This ratio is equal to 1 only if $\theta_a = \theta_b$. So, if $\theta_a \neq \theta_b$, the assignment function $\alpha(b) = b$ could not be an equilibrium object.

This result is actually more general as the following remark states.

Remark 10 *Assume $H(.) = G(.)$ and symmetric production function $f(x,y) = f(y,x)$ then $\frac{\omega(x)+\eta}{\varpi(\alpha(x))+\eta} = 1$ for all values of b, and the assignment function is $\alpha(b) = H^{-1}(G(b))$ independent of c and λ.*

The idea is that in the symmetric case the inaction are also symmetric. Then, in general, $c > 0$ will change the shape of the assignment function. The way the assignment function will change depends on how the functions $\varpi(a)$ and $\omega(b)$ change with c.

6.3 Steady state distribution and endogenous mismatch

The steady state distribution of matches (a,b) is as follows. Let

$$\sigma(a,b)=\begin{cases} h(a)\,g(b) & b\in\mathbf{B}(a) \\ 0 & otherwise \end{cases}, \tag{13}$$

and

$$\Sigma(b|a)=\begin{cases} \int_{b_0}^{b}\sigma(x,y)\,dxdy & b<\beta(a) \\ h(a)\left[\Pr\left[b\notin\mathbf{B}(a)\right]\right]+\int_{b_0}^{b}\sigma(a,y)\,dxdy & b\geq\beta(a) \end{cases}. \tag{14}$$

Then, the mass of matches with talent less than a and productivity less than b is given by

$$\Sigma(a,b)=\int_{a_0}^{a}\Sigma(b|x)\,dx \tag{15}$$

It is easy to see that in equilibrium the actual productive pairs (a,b) will not be positive assortative, in the sense that there will be high and low values of a matched with the same productivity type b. The level of mismatch will depend on the size of the inaction bands and

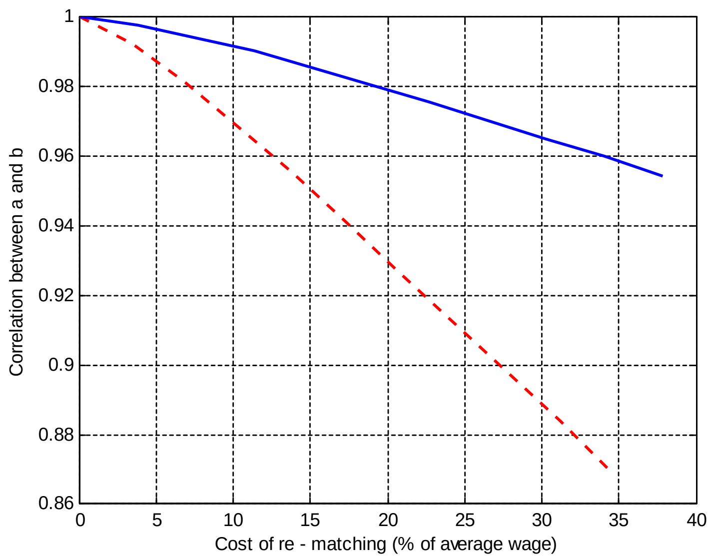

Figure 3: **Mismatch.**
This figure depicts the correlation between a and b for different values of the re-matching cost (measured as % of average payment of the agent) for uniform distributions functions $H(.)$ and $G(.)$ with support $[0, 0.9]$ and production function $f(a,b) = \frac{a}{1-b}$. The continuous blue line is drawn for $\lambda = 0.1$ while the dashed red line is drawn for $\lambda = 0.3$. Results are based in a Montecarlo simulation.

therefore on the elements that modified them (see section 6.1). To get a sense of the degree of the misallocation figure 3 shows the correlation between a and b for different values of the re-matching cost and persistence of the shock. Of course, if $c = 0$ there is no mismatch and the correlation is one for any value of λ. As the cost of re-matching increases, the inaction band becomes wider and the correlation decreases. The effect is larger, the lower the persistence of the shock. When $\lambda = 0.1$ (the blue line in figure 3) the effect of higher values of c is smaller, reducing the correlation from 100% to around 95% when the cost of re-matching is around 37% of the average per period payment of the entrepreneur. In the case of lower persistence, $\lambda = 0.3$, the effect is more pronounced as shown by the red line: the correlation goes down to 87% when the cost of re-match goes to 35% of the value of the average entrepreneur payment.

6.4 Factor payment distribution

Finally, I show the effect of the re-matching cost on factor payment distributions, where the payments are computed as the per period flows in section 5. As a benchmark I consider the factor payment distribution when $c = 0$. Figure 4 shows the histogram of the entrepreneurs' wage (relative to the wage when $c = 0$). Part (i) of this figure is the distribution conditional on $a = 0.45$ and part (ii) is the wage distribution conditioned on $b = 0.45$. In both cases the black filled bars are for the benchmark case, and the unfilled bars represent the case where $c = 0.3$.

Starting with entreprenuers payments. In the benchmark case, since everybody is matched

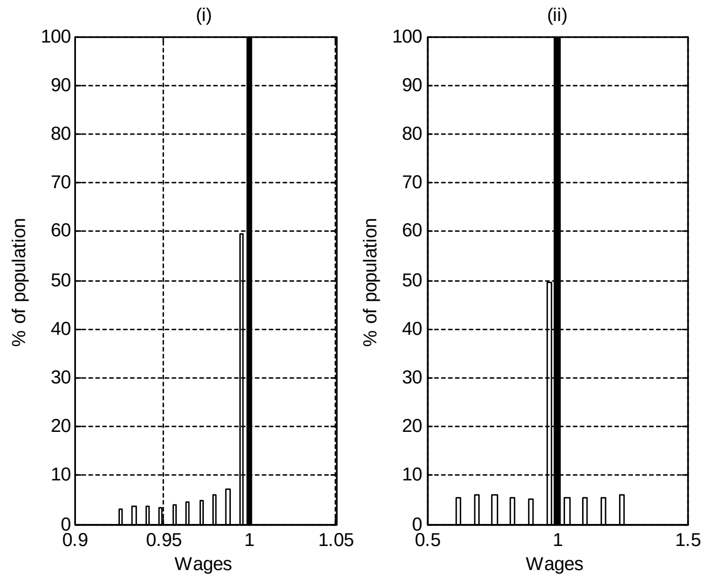

Figure 4: **(Conditional) wage distribution.**
These figures depict wage histograms (relative to $c = 0$) conditional on $a = 0.45$ - part (i) - and conditional on $b = 0.45$ - part (ii) - for uniform distributions functions $H(.)$ and $G(.)$ with support $[0, 0.9]$, production function $f(a, b) = \frac{a}{1-b}$ and $\lambda = 0.3$. White bars are drawn for $c = 0.3$ and black ones for $c = 0$. Results are based on a Montecarlo simulation.

with his perfect mate, the conditional distribution is concentrated in one point. In contrast, when $c = 0.3$, there is endogenous mismatch and so the distribution of wages is not concentrated in a point. In particular, conditional on talent $a = 0.45$ (part i), the majority of entrepreneurs are matched with the ideal firm (60% of this type of entrepreneurs are in this situation). The rest get a lower payment since they are working for firm different from the optimal. The reduction in the payment is, of course, limited by the ability of entrepreneurs to sell the firm. Part (ii) presents the same exercise but fixing the type of firm at $b = 0.45$. The largest portion of entrepreneurs managing this type of firm, around 50%, has the ideal level of talent for a firm with productivity $b = 0.45$ ($\alpha(0.45)$) and gets a payment similar to the equilibrium one in the case of no cost of re-matching. The difference occurs because the assignment function changes a little bit and because the value of a match is smaller because the cost of re-matching is higher. With costly re-matching, there are entrepreneurs managing this type of firm that have more or less talent than the "ideal" one, and so some of them are earning more and others are earning less than the "ideal" one.

Figure (5) presents the same exercise for the distribution of firm value[9]. Conditional on $a = 0.45$, the distribution of firm values is, of course, concentrated in one point when $c = 0$ and dispersed when $c = 0.3$, because in the latter case there are many types of firms own by the entrepreneur with $a = 0.45$. That is, entrepreneurs with the same ability are managing firms with different values.

[9]Figure (5) does not show the histrogram of firm values conditional on b, since the value of the firm conditional on b is the same for all a.

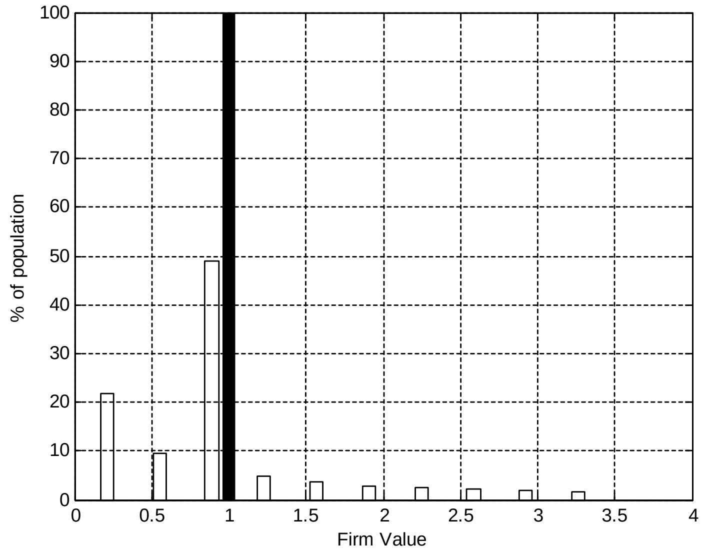

Figure 5: **(Conditional) firm value distribution.** This figure depicts firm value histogram (relative to $c = 0$) conditional on $a = 0.45$ for uniform distributions functions $H(.)$ and $G(.)$ with support $[0, 0.9]$, production function $f(a, b) = \frac{a}{1-b}$ and $\lambda = 0.3$. White bars are drawn for $c = 0.3$ and black ones for $c = 0$. Results are based on a Montecarlo simulation.

Finally, figure (6) shows the (unconditional) distribution of wages and firm values for $c = 0$ (the black filled bars) and for $c = 0.3$ (the unfilled bars). Starting with panel (ii), note that the distribution of firms value is not affected in any significant way when the cost of re-matching increases from 0 to 0.3. This occurs because, by assuming that all the cost of re-match is paid by the entrepreneur, the value of the firm does not depend on which entrepreneur is managing the firm. In contrast, the distribution of entrepreneur payments changes a lot when c is increased from 0 to 0.3. As could be seen in panel (i), entrepreneur payments are more dispersed and lower on average when the cost of re-matching increases. This is a direct consequence of the misallocation of entrepreneurs to firms.

7 Extensions

In this section I present two extensions to the basic model. First, I introduce a tenure effect into the production function. The idea is that as time passes and the sides of the match know each other better, productivity increases. This match-specific component has no value outside the match and so makes starting a new partnership less attractive as time passes, reducing the turnover with tenure, a fact that is common in many applications. The second extension generalize the stochastic process in (1) by letting the new productivity level depend on the type of entrepreneur that is managing the firm. The idea is that more talent entrepreneurs have a higher probability of getting a higher productivity draw than less talent ones.

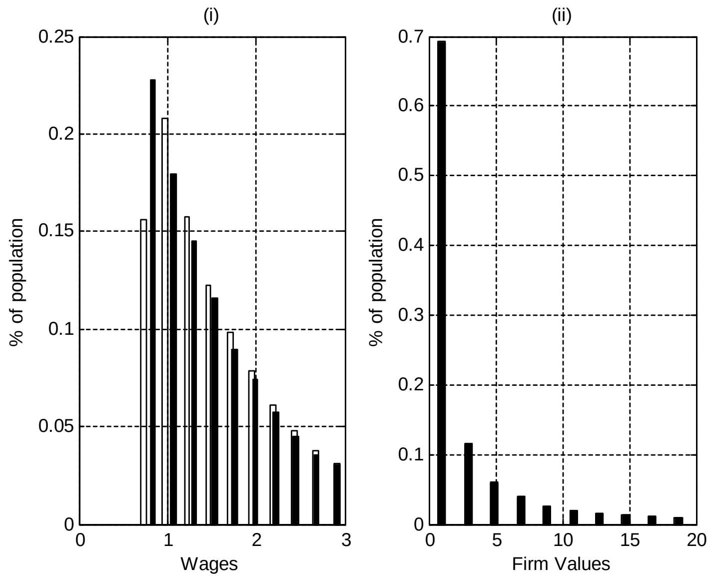

Figure 6: **Wage and firm value distribution.**
These figures depict wage - part (i)) - and firm value - part (ii) - histograms for uniform distributions functions $H(.)$ and $G(.)$ with support $[0, 0.9]$, production function $f(a,b) = \frac{a}{1-b}$ and $\lambda = 0.3$. White bars are drawn for $c = 0.3$ and black ones for $c = 0$. Results are based on a Montecarlo simulation.

7.1 Tenure effects

In this section I consider a production function that is increasing in the duration of the match. More formally,

Assumption 11 (Production function with tenure effect) *Let τ be the time a partnership has been producing together, the production function is given by $f(a,b,\tau)$ with $f_\tau(a,b,\tau) > 0$ and $f_{a,b}(a,b,\tau) > 0$ for all τ.*

The match value function is now given by,

$$rM(a,b,\tau) = \left\{ \begin{array}{c} f(a,b,\tau) + M_\tau(a,b,\tau) \\ +\lambda\left\{\int \max\left[M(a,b',\tau), W(a) + V(b')\right] G(db') - M(a,b,\tau)\right\} \\ +\eta\left\{W(a) + V(b) - M(a,b,\tau)\right\} \end{array} \right\}. \tag{16}$$

Besides the dependence on tenure, the major difference with respect to the value function in equation (12) is that there is a new term, $M_\tau(a,b,\tau)$, that reflects the effect of seniority.

In order to use all the material from the basic model, it is useful to work with production function that could be written as follows $f(a,b,\tau) = f^1(b,a) + f^2(a,\tau)$. A simple inspection of the value function above shows that in this case $M(a,b,\tau) = M^1(b,a) + M^2(a,\tau)$. Under this assumption, the supermodularity of the match value function follows trivially.

As before, $M_b(a,b,\tau) = \frac{f_b(a,b,\tau)}{r+\lambda+\eta}$ and then the market value of a firm is,

$$V(b) = \int_{b_0}^{b} \frac{f_b(\alpha(x), x, 0)}{r+\lambda+\eta} dx. \tag{17}$$

Given this, the equation that define the boundaries of the inaction bands is given by,

$$\frac{1}{r+\lambda+\eta}\left\{f(a,b,\tau)-f(a,\beta(a),0)-\int_{\beta(a)}^{b}f_b(\alpha(x),x,0)\,dx\right\}=-c.$$

Since $f(a,b,\tau)=f^1(b,a)+f^2(a,\tau)$, this equation could be written as follows,

$$\frac{1}{r+\lambda+\eta}\left\{\begin{array}{l} f^1(b,a)-f^1(a,\beta(a)) \\ -\int_{\beta(a)}^{b}f_b^1(\alpha(x),x)\,dx \end{array}.\right\}=-\left(c+f^2(a,\tau)-f^2(a,0)\right)$$

since $f_\tau^2(a,\tau)>0$, having a tenure effect is equivalent to having a setup cost that is increasing with the tenure. Therefore, for any given pair (a,b) the inaction bands increase with tenure and so match turnover decreases with time.

Finally, note that the aggregate demand and supply of firms are given by

$$S(b)=\int_0^\infty\int_{b_L}^{b}g(x)\left[\omega(x,\tau)+\eta\right]dxd\tau.$$

$$D(a)=\int_0^\infty\int_{a_L}^{\alpha(a)}h(\alpha(x))\left[\varpi(\alpha(x),\tau)+\eta\right]dxd\tau.$$

and then the clearing market condition in equation (8) is still valid.

7.2 State dependent shock

One of the implications of the stochastic process in assumption 1 is that after being hit by a shock the new productivity level of the firm is independent of both the quality of the entrepreneur in charge of the firm and the previous productivity level of the firm. In this section I relax the first assumption. In particular let me replace $G(.)$ by $G(.|a)$ in assumption 1.

The key issue is to verify that the match value function is still super modular. To see that this is indeed the case note that, as in section 2, the match value function is just

$$(r+\lambda+\eta)\,M(a,b)=f(a,b)+\lambda\Phi(a). \tag{18}$$

where

$$\Phi(a)=\int\left\{\max\left[M(a,b'),W(a)+V(b')\right]G(db'|a)\right\}+\eta\left\{W(a)+V(b)\right\}.$$

then $M_{a,b}(a,b)=\frac{f_{a,b}(a,b)}{r+\lambda}>0$, and, as before, the match value function is supermodular.

The assignment function will be different, since a new source of interaction between characteristics a and b has been added. Let $g(b|a)=G_b(b|a)$ and $\overline{g}(b)=\int_{a_0}^{a_1}g(b|a)\,h(a)\,da$, then the supply of firms with productivity less or equal to b is given by

$$S(b)=\lambda\int_{b_0}^{b}\overline{g}(x)\left[\omega(x)+\eta\right]dx.$$

where as before the function $\omega(b)=\lambda\Pr\left[a\notin\mathbf{A}(b)\right]$ is the intensity at which at which firms with productivity level b go tothe market. The supply of firms is still given by equation (6) and the equilibrium condition is again given by equation (8).

8 Conclusions

This paper presents a tractable assignment model that allows partners' characteristics to change over time. This extension is important, since it extends assignment models in a way that permit the study of the dynamics of the matching process and the possibility of equilibrium mismatch. It is not difficult to imagine situations where this time variability of partners' characteristics is relevant and for which the static nature of previous assignment models was a limitation. I hope the model presented here will be a useful framework in which to study this type of problem.

One of the main contributions of the paper is to state conditions under which the model is tractable. The paper contains proof of existence of the equilibrium, as well as an algorithm to compute it. The paper also includes some extensions that could be important for empirical applications. In particular, study the effect of match-specific components, nesting traditional assignments models in the style of Sattinger (1979) where partners characteristics could be traded in a competitive market with papers in the line of Jovanovic (1979) that exploit the effect of productivity components useful only inside the match, but with no value outside.

Several extensions would enrich the analysis. In particular, a more general specification of the stochastic process would be useful if one would like to use the model in context of learning.

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A Two sides stochastic types

As was pointed out in the text, the key element for existence of an equilibrium characterized by (i) a positive assortative matching between entrepreneur types and firm productivity levels in the market for firms and (ii) possible non assortative matching between firms and entrepreneurs that decide not to go to the market is the supermodularity of the match-value function. Here, I show that this is the case when both sides' types are stochastic.

Assumption 12 *Given any current entrepreneur type a, with intensity λ_a the productivity level changes to $a' \sim H(.)$, where $H(.)$ is a distribution function with support in $[a_L, a_H]$ and with density function $0 < h(.) < \infty$ for all a in $[a_L, a_H]$.*

The match value function is now given by

$$rM(a,b) = \left\{ \begin{array}{c} f(a,b) + \lambda_a \left\{ \int \max\left[M(a',b), W(a') + V(b)\right] H(da') - M(a,b) \right\} \\ +\lambda_b \left\{ \int \max\left[M(a,b'), W(a) + V(b')\right] G(db') - M(a,b) \right\} \\ +\eta \left\{ W(a) + V(b) - M(a,b) \right\} \end{array} \right\}, \quad (19)$$

where λ_b replaces λ in the main text.

Proposition 13 *$M(a,b)$ is supermodular.*
Proof. *It follows directly from equation (19), since the cross derivative is $\frac{f_{ab}(a,b)}{r+\lambda_a+\lambda_b+\eta} > 0$ by assumption 3* ■

B Proof of proposition (7)

As shown in section (4), the assignment function $\alpha(.)$ is a fixed point of the following functional equation (equation(10) in the main text):

$$\alpha(b) = a_0 + \int_{b_0}^{b} H(x, \alpha(x); \alpha)\, dx, \quad (20)$$

where[10] $H(x, \alpha(x); \alpha) = \frac{g(x)[\omega(x;\alpha)+\eta]}{h(\alpha(x))[\varpi(\alpha(x);\alpha)+\eta]}$ and $\varpi(a;\alpha) = \lambda \Pr[b \notin \mathbf{B}(a;\alpha)]$ and $\omega(b;\alpha) = \lambda \Pr[a \notin \mathbf{A}(b;\alpha)]$.

To find the solution to this problem, I study the following operator

$$T(\alpha)(b) := a_0 + \int_{b_0}^{b} \widehat{H}(x, \alpha(x); \alpha)\, dx, \quad (21)$$

where $\widehat{H}(x, \alpha(x); \alpha) = \frac{a_1 - a_0}{\varkappa(\alpha)} H(x, \alpha(x); \alpha)$ and $\varkappa(\alpha) = \int_{b_0}^{b_1} H(x, \alpha(x); \alpha)\, dx$, and show that its solutions coincide with those of (20). In what follows, I let M to be the set of bounded increasing bijective continuous functions mapping $[b_L, b_H]$ onto $[a_L, a_H]$.

[10] For any function F the notation $F(x, \alpha(x); \alpha)$ is used to emphasized that F depends on both the assignment function evaluated at some point $(\alpha(x))$ and on the assignment function itself. (α)

Preliminaries Before studying this problem, I present some lemmas that are going to be used afterwards.

Lemma 14 *T is equicontinuous.*

Proof: Consider an arbitrary $\alpha \in M$ and an arbitrary $b \in [b_L, b_H]$. First, note that $\widehat{H}(b, \alpha(b); \alpha)$ is bounded above and below. In particular, since:

1. Functions g and h are bounded
$$\begin{array}{ll} p_g \le g(b) \le P_g & for \;\; all \;\; b \in [b_L, b_H], \\ p_h \le h(a) \le P_h & for \;\; all \;\; a \in [a_L, a_H]. \end{array}$$
2. And
$$\begin{array}{ll} 0 \le \lambda \Pr[a \notin \mathbf{A}(x;\alpha)] \le \lambda & for \;\; all \;\; b \in [b_L, b_H], \\ 0 \le \lambda \Pr[b \notin \mathbf{B}(x;\alpha)] \le \lambda & for \;\; all \;\; a \in [a_L, a_H]. \end{array}$$
3. And $\eta > 0$.

$\widehat{H}(x, \alpha(x); \alpha)$ is bounded above and below by

$$\frac{a_1 - a_0}{b_1 - b_0} \frac{p_h p_g}{P_g P_h} \left(\frac{\eta}{\lambda + \eta}\right)^2 \le \widehat{H}(x, \alpha(x); \alpha) \le \frac{a_1 - a_0}{b_1 - b_0} \frac{P_h P_g}{p_h p_g} \left(\frac{\lambda + \eta}{\eta}\right)^2,$$

Second, since $\left|\frac{\partial}{\partial b}(T\alpha)(b)\right| = \left|\widehat{H}(b, \alpha(b); \alpha)\right|$, it has to be the case that

$$|(T\alpha)(b) - (T\alpha)(b')| \le K|b - b'|,$$

where

$$K = \left|\frac{a_H - a_L}{b_H - b_L} \frac{P_h P_g}{p_h p_g} \left(\frac{\lambda + \eta}{\eta}\right)^2\right|,$$

Then, for every $\varepsilon > 0$, there is a $\gamma = \frac{\varepsilon}{K}$ such that $|b - b'| < \gamma$ implies $|(T\alpha)(b) - (T\alpha)(b')| < \varepsilon$. Since this was done for an arbitrary function $\alpha \in M$ and $b \in [b_L, b_H]$, it follows that T is equicontinuous.

Lemma 15 *The boundaries of the inaction sets are continuous in α*

Proof: I consider the boundaries of the set $\mathbf{B}(a; \alpha(.))$ of acceptable firms types for an entrepreneur with ability a. The arguments are the same for the boundaries of the set $\mathbf{A}(b; \alpha(.))$ of entrepreneurs talent that in equilibrium may be working with a type b firm.

Given a and $\alpha(.)$, the boundaries of the inaction set $\mathbf{B}(a; \alpha(.))$, $\underline{b}(a; \alpha(.))$ and $\overline{b}(a; \alpha(.))$, are the solution to $I(a, b; \alpha) = W(a, b) - W(a, \gamma(a)) = 0$. Using equations () and (), this equation could be written in the following way.

$$I(a, b; \alpha) = f(a, b) - f(a, \gamma(a)) - \int_{\gamma(a)}^{b} f_b(\alpha(x), x)\, dx + c = 0,$$

where the function $\gamma(a) = \alpha^{-1}(a)$, mapping $[a_0, a_1]$ onto $[b_0, b_1]$, is well defined, since α is bijective.

It was already shown that under assumption 3 in the text (supermodularity of the production function) for all $\alpha \in M$ this equation has two solutions when both $I(a, b_0; \alpha) < 0$ and $I(a, b_1; \alpha) < 0$, one solution when either $I(a, b_0; \alpha) < 0$ and $I(a, b_1; \alpha) \geq 0$ or viceversa, and no solution otherwise. The boundaries $\underline{b}(a; \alpha(.))$ and $\bar{b}(a; \alpha(.))$ are defined as follows

$$\begin{aligned} \underline{b}(a; \alpha(.)) &= \begin{cases} b_0 & if \;\; I(a, b_0; \alpha(.)) \geq 0 \\ smallest \;\; solution \;\; of \; I(a, b; \alpha(.)) = 0 & otherwise \end{cases}, \\ \bar{b}(a; \alpha(.)) &= \begin{cases} b_1 & if \; I(a, b_1; \alpha(.)) \geq 0 \\ largest \; solution \; of \; I(a, b; \alpha(.)) = 0 & otherwise \end{cases}. \end{aligned}$$

The function $I(a, b; \alpha)$ is continuous in b, and since b takes values in a compact set it is also uniformly continuous on this set.

Second, given a and b, $I(a, b; \alpha(.))$ is also continuous in α. That is

$$\|I(a, b; \alpha_i) - I(a, b; \alpha)\| \to 0 \;, \quad i \to \infty,$$

where $\|.\|$ is the symbol for the sup-norm.

To see this, consider a sequence $\{\alpha_i\}$ in M converging in the sup norm to α. Note that

$$I(a, b; \alpha_i) - I(a, b; \alpha) = Q_1 + Q_2 + Q_3,$$

where

$$\begin{aligned} Q_1 &= -f(a, \gamma_i(a)) + f(a, \gamma(a)), \\ Q_2 &= \int_{\gamma(a)}^{\gamma_i(a)} f_b(\alpha_i(x), x)\, dx, \\ Q_3 &= \int_{\gamma(a)}^{b} [f_b(\alpha(x), x) - f_b(\alpha_i(x), x)]\, dx, \end{aligned}$$

then,

$$\|I(a, b; \alpha_i) - I(a, b; \alpha(.))\| \leq \|Q_1\| + \|Q_4\| + \|Q_3\|.$$

Clearly, $\|Q_1\|, \|Q_2\| \to 0$ when $i \to \infty$. It is also the case that $\|Q_3\| \to 0$, since $f_b(.)$ is continuous, so it is uniformly continuous in the relevant set, and because α_i converge to α in the sup norm.

Finally, consider a sequence $\{\alpha_i\}$ in M converging in the sup norm to α, and the corresponding sequence $\{\gamma_i\}$ Define the sequence $\{b_i\}$ as the lower boundary consistent with $\alpha_i(.)$. Let $\underline{b}^*$ be the lower boundary when the assignment function is $\alpha(.)$, and note that as $\alpha_i \to \alpha$ (and $\gamma_i \to \gamma$) it should be the case that $b_i \to b^*$, since $I(a, b_i; \alpha_i)$ is uniformly continuous in all its arguments.

Corollary 16 *$\widehat{H}(x, \alpha(x); \alpha)$ is continuous in α.*

Proof: This is the case since α affects $\widehat{H}(x,\alpha(x);\alpha)$ directly evaluated at $\alpha(x)$ througth continuos functions, and indirectly througth the effects of α in the boundaries of the inaction set. But from lemma 15 the boundaries are continuous on α, and $\widehat{H}(x,\alpha(x);\alpha)$ is a continuous function of the boundaries.

Lemma 17 *T is continuos as an operator.*

Proof: Consider a sequence $\{\alpha_i\}$ in M converging in the sup norm to α, then

$$\|(T\alpha_i)(b)-(T\alpha)(b)\| \leq (b-b_0)\max\left\{\begin{array}{c}\left|\widehat{H}(x,\alpha_i(x);\alpha)-\widehat{H}(x,\alpha(x);\alpha)\right| \\ : x\in[b_0,b]\end{array}\right\}.$$

Continuity needs

$$\lim_{i\to\infty}\max\left\{\left|\widehat{H}(x,\alpha_i(x);\alpha)-\widehat{H}(x,\alpha(x);\alpha)\right| : x\in[b_0,b]\right\}=0$$

.If this where not the case, there would be a $\gamma>0$ and a sequence of $\{x_i\}$ in $[b_0,b]$ such that

$$\left|\widehat{H}(x,\alpha_i(x_i);\alpha)-\widehat{H}(x,\alpha(x);\alpha)\right| \geq \gamma\ , \qquad i=1,2,... \tag{22}$$

But $\{x_i\}$ has a subsequence in $[b_0,b]$ that converges. Let denote this subsequence again by $\{x_i\}$ and denote $x^*=\lim_{i\to\infty}x_i$. Since $\|\alpha_i-\alpha\|\to 0$

$$|\alpha_i(x_i)-\alpha(x^*)| \leq |\alpha_i(x_i)-\alpha(x_i)|+|\alpha(x_i)-\alpha(x^*)|\to 0,$$

then $\alpha_i(x_i)\to\alpha(x^*)$, but since both α and $\widehat{H}$ are continuous $\lim_{i\to\infty}\widehat{H}(x^*,\alpha_i(x_i)) = \widehat{H}(x^*,\alpha(x^*)) = \lim_{i\to\infty}\widehat{H}(x^*,\alpha(x_i))$, contradicting (22).

Proof of proposition 7 (existence) I used the following generalization of the Schauder fixed point theorem (Ok (2007), p.629, excersice 38)

Theorem 18 (Generalization of Schauder fixed point theorem) *Let S be an nonempty bounded and convex subset of a normed linear space X, and let Φ be a continuous self-map on S such that $cl_X(\Phi(S))$ is compact, then Φ has a fixed point.*

To apply this theorem, consider the normed linear space $Q([b_L,b_H])$ of all bounded continuous real functions defined on the compact set $[b_L,b_H]$ with the sup norm and let M be the set of bounded increasing bijective continuous functions that map $[b_L,b_H]$ onto $[a_L,a_H]$. Note that M is a nonempty bounded and convex subset of $Q([b_L,b_H])$. From lemma 17 T is a continuous self-map on M. To see that $cl_{Q([b_L,b_H])}(T(M))$ is compact, note that since $T(M)$ is bounded (it is contained in M), the Aezelà-Ascoli theorem says that all I need to show is that $T(M)$ is equicontinuous. But this was already shown in lemma 14. Therefore, the operator T has at least one fixed point.

Finally, note that $\alpha(b_1) = a_1$ and $\alpha(b_0) = a_0$, since

$$\alpha(b_1) - \alpha(b_0) = \int_{b_0}^{b_1} \widehat{H}(x, \alpha(x))\, dx,$$

it should be the case that $\int_{b_0}^{b_1} \widehat{H}(x, \alpha(x))\, dx = a_1 - a_0$ and therefore $\alpha(b)$ also solves equation (20)

$$\alpha(b) = a_0 + \int_{b_0}^{b} H(x, \alpha(x))\, dx.$$

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